Naturality and a Universal Property for Polynomial Functors

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Motivation: Classifying Spaces

1850s-
1900s

- Topological features
- Combinatorial invariants

Examples:

Path-connected, compact, Hausdorff, second countable, locally Euclidean, simply connected, etc.

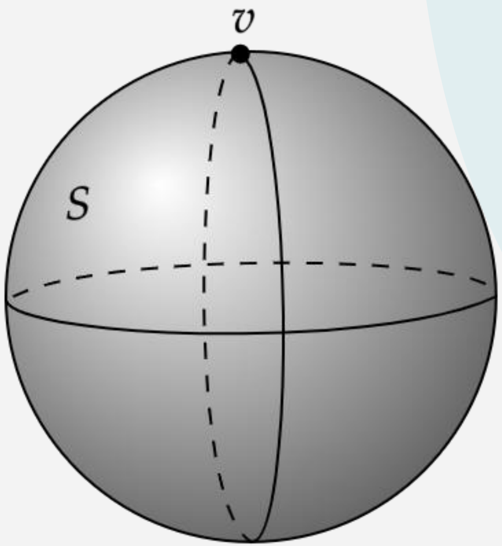


Fig 1. A hollow sphere with surface S and specified point v.

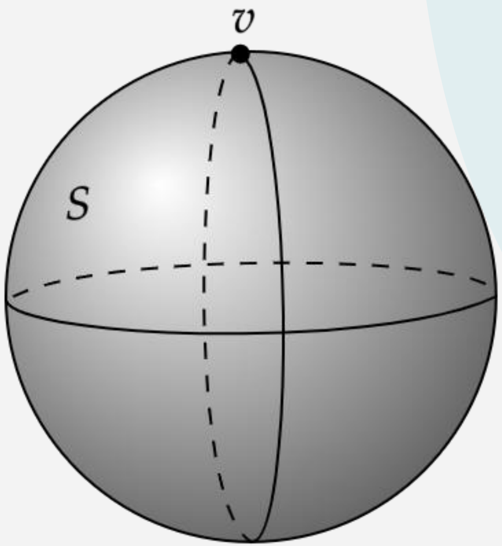
Motivation: Classifying Spaces

1920s-present

- Algebraic invariants
- Functorial invariants

Simplicial-chain for S^2: [1]

$$C^{\Delta}(S^2) = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_S \longrightarrow 0 \longrightarrow \mathbb{Z}_v$$



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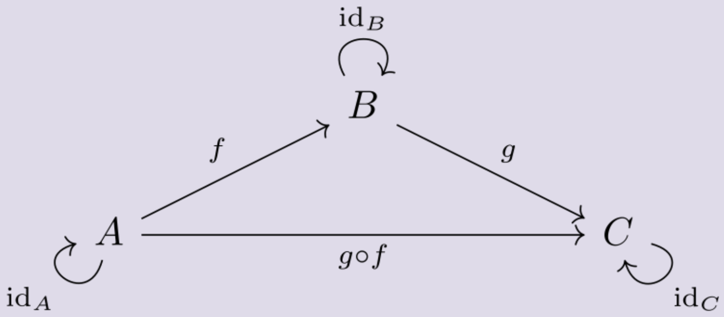
Categories and Functors

Defn: Categories [2]

A category, $\mathcal{C}$, consists of a collection of objects and maps between objects which can be composed:



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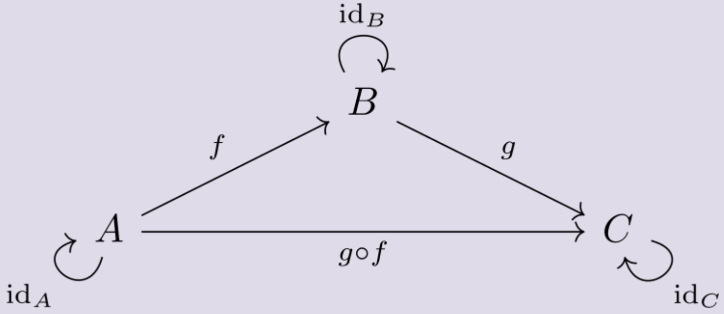
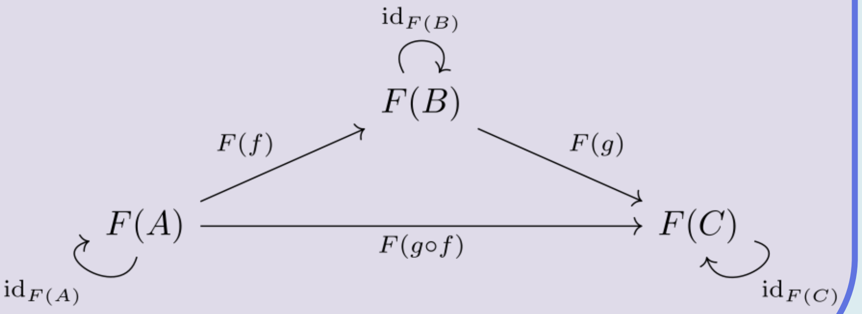


Defn: Functor [2]

A functor, $F : \mathcal{C} \to \mathcal{D}$, is a function on objects and maps that preserves commuting diagrams:



Naturality

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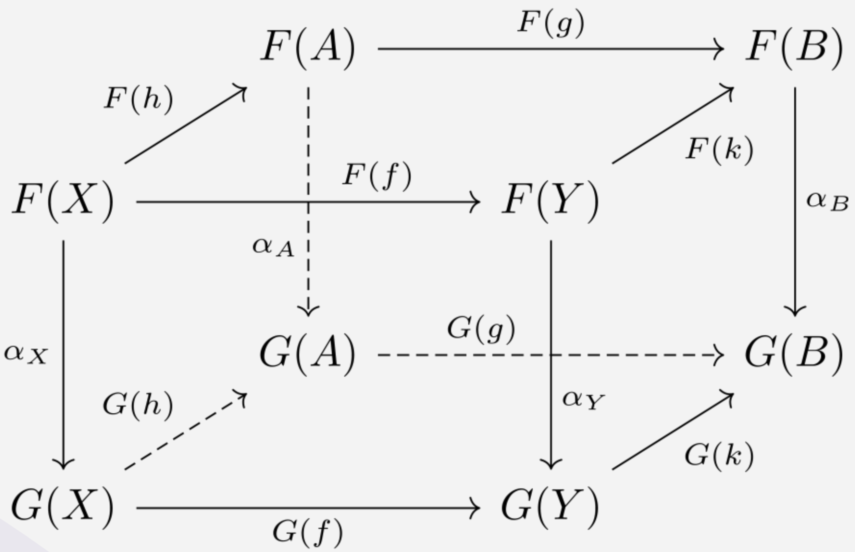

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Polynomial Approximations

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Question: What do we do for functions $f\colon \mathbb{R} \to \mathbb{R}$?

$$e^x \sim 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

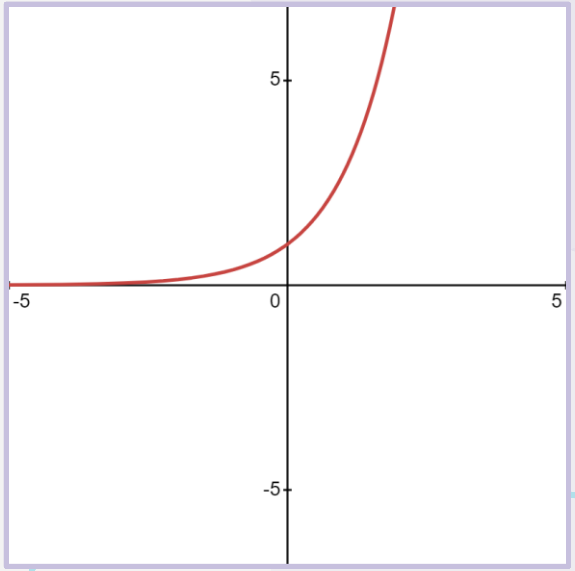

Fig 2.a Exponential function

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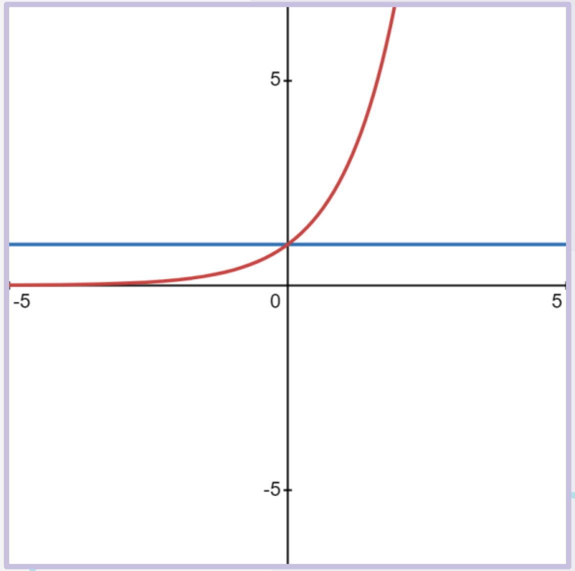

Fig 2.b e^x to 0th-degree

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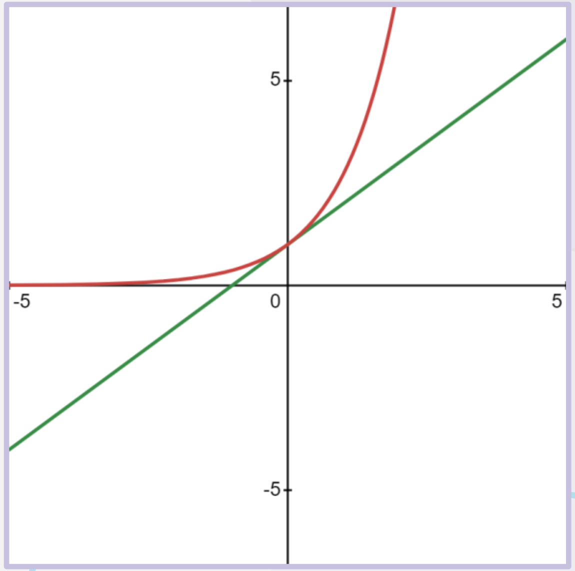

Fig 2.c e^x to 1st-degree

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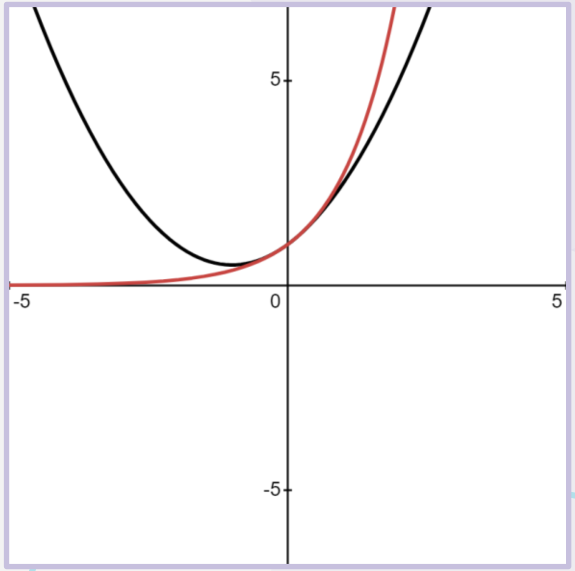

Fig 2.d e^x to 2^{nd}-degree

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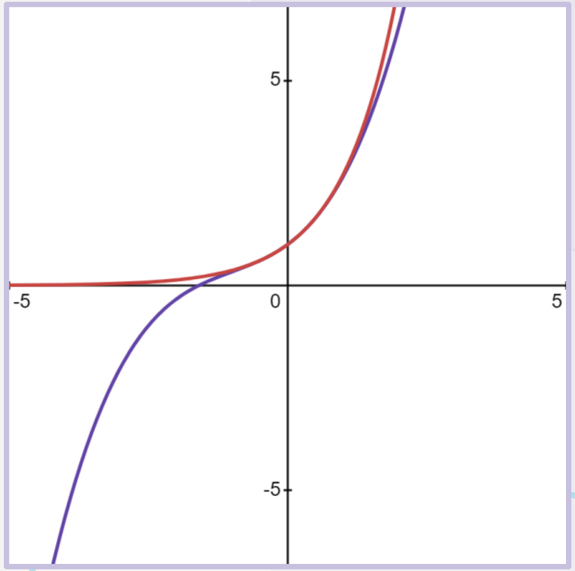
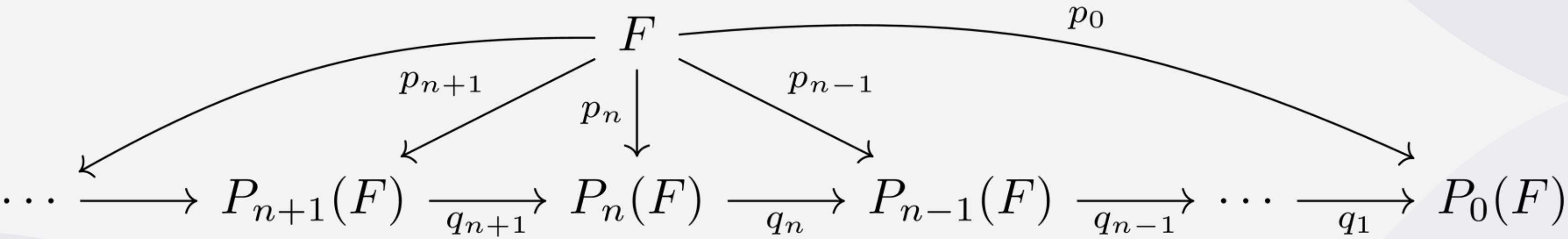


Fig 2.e e^x to 3rd-degree

Polynomial Approximations

Idea: We want to study the functor $F\colon \mathcal{B} \to Ch(Ab)$ a sequence of simpler functors $P_n(F)\colon \mathcal{B} \to Ch(Ab), n \geq 0$, which approximate F in a universal, but homotopical, fashion. [3,4]



How do we construct polynomial approximations $P_n(F)$?

Cross Effects: Measuring Defects

Polynomial Defects:

For $f : \mathbb{R} \to \mathbb{R}$, the defect to f being polynomial can be measured inductively:

$$\mathrm{cr}_1(f)(x) = f(x) - f(0)$$

$$\mathrm{cr}_2(f)(x, y) = \mathrm{cr}_1(f)(x + y) - \mathrm{cr}_1(f)(x) - \mathrm{cr}_1(f)(y)$$

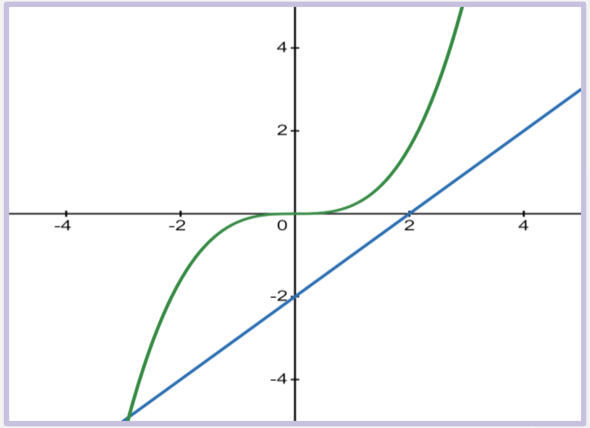


Fig 3. Cubic and linear function

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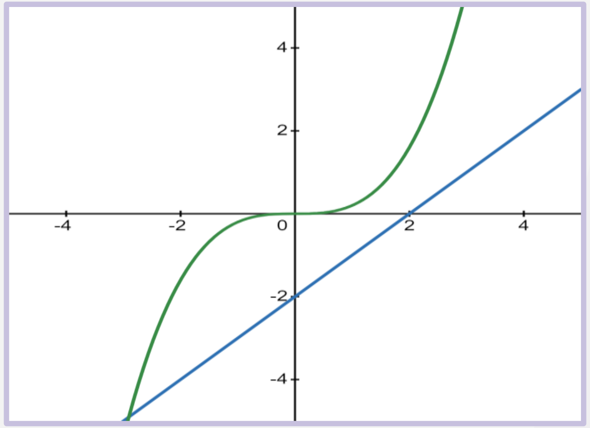

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Remark: Eilenberg and MacLane in [1] generalized cross-effects to functors valued in categories with direct sums, $F\colon \mathcal{B} \to \mathcal{A}$:

$$\mathrm{cr}_1(F)(A) \oplus F(0) \cong F(A)$$

$$\mathrm{cr}_2(F)(A,B) \oplus \mathrm{cr}_1(F)(A) \oplus \mathrm{cr}_1(F)(B) \cong \mathrm{cr}_1(F)(A \oplus B)$$

Affine Example:

Example: "f(x) = x + a"

Let $A \in \mathsf{Ab}$ and let $T_A : \mathsf{Ab} \to \mathsf{Ch}(\mathsf{Ab})$ be given by $T_A(B) = \cdots \to 0 \to 0 \to A \oplus B$. Then

$$\mathrm{cr}_1(F)(B) \cong \cdots \to 0 \to 0 \to B$$

and

$$\mathrm{cr}_2(T_A)(B, C) \cong \cdots \to 0 \to 0 \to 0$$

Construction:

(1) For an invariant F, the cross-effect gives $C_{n+1}(F)$, $C_{n+1}(F)(B) := \mathrm{cr}_{n+1}(F)(B, ..., B)$, which measures nth-degree defects.

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(2) For $P_n(F)$ we resolve F with respect to the defects $C_n(F)$:

$$\cdots \to C^3_{n+1}(F) \to C^2_{n+1}(F) \to C_{n+1}(F) \to F$$

and then totalize.

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(3) The universal approximating map $p_n : F \to P_n(F)$ is given by including F into its C_{n+1} resolution before totalizing.

Affine Example:

Example: "f(x) = x + a"

For $T_A : \mathsf{Ab} \to \mathsf{Ch}(\mathsf{Ab})$,

$$P_0(T_A)(B) = \cdots \to B \xrightarrow{\mathrm{id}_B} B \xrightarrow{0} B \xrightarrow{i} A \oplus B$$

After contracting:

$$P_0(T_A)(B) \simeq_{nat} \cdots \to 0 \to 0 \to 0 \to A$$

Universality

Slogan: $P_n(F)$ universally approximates F, up to **natural** homotopy.

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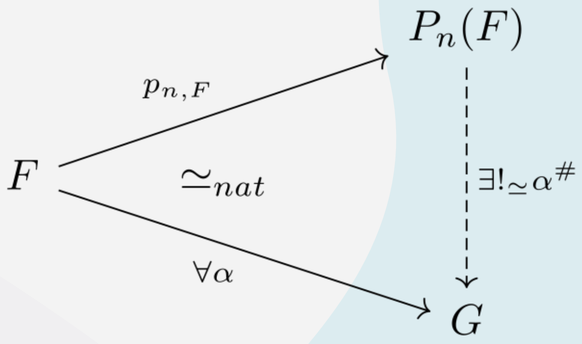
Thm: Universal Degree n Approximation

Let $F : \mathcal{B} \to \mathsf{Ch}(\mathsf{Ab})$ be an invariant. Then:

(i) The functor $\mathrm{cr}_{n+1}(P_n(F))$ is **naturally** contractible.

(ii) $p_{n,F} : F \to P_n(F)$ is universal up to **natural** homotopy among degree n maps.



Key Takeaways:

Algebraic Invariants

Powerful tool for classifying spaces, but rich and complicated

Polynomial Approximation

Provides a Taylor series-like approach to studying algebraic invariants

Naturality

Improves coherency of universal homotopies with respect to commutative diagrams and allows for extensions to infinity-categories

Thank you!

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References:

[1] S. Eilenberg and S. MacLane. "On the Groups $H(\Pi, n)$,II: Methods of Computation". In: *Annals of Mathematics* 60.1 (1954), pp. 49-139.

[2] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017.

[3] B. Johnson and R. McCarthy."Deriving calculus with cotriples". In: *Transactions of the American Mathematical Society* 356.2 (2004), pp. 757-803.

[4] T. G. Goodwillie, "Calculus III: Taylor Series". In: *Geometry & Topology* 7 (2003), pp. 645-711.

[5] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[6] C. A. Weibel. "CHAPTER 28 – History of Homological Algebra". In: *History of Topology*. Ed. By I. James. Amsterdam: North-Holland, 1999, pp. 797-836.