Applications and Examples in Equivariant Poincare Duality

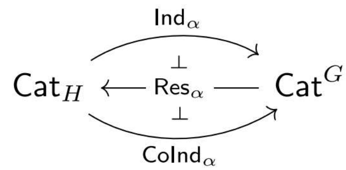
Introduction/Motivation

These notes are for a 30~40 minute talk on applications and examples of twisted ambidexerity in equivariant Poincare duality in Sections 4.1-4.4 of the paper *Parametrised Poincare Duality and Equivariant Fixed Points Methods*[1], which was given as part of an Ambidexterity seminar at UIUC in Fall 2025.

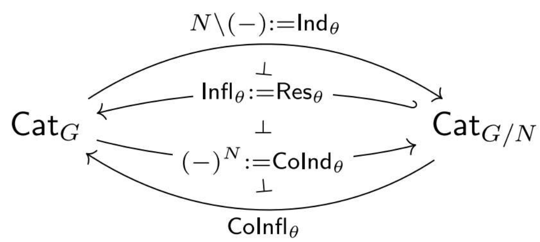
Let's begin by recalling some preliminary notions and notations in the parameterized formulation of equivariant homotopy theory. Throughout G will be a fixed compact Lie group. We write $\mathcal{S}^G := \mathsf{PSh}(\mathsf{Orb}^G)$ for the category of **G-spaces** where Orb^G is the full subcategory of $\mathcal{S}^{BG}$ spanned by the orbits G/H for $H \leq G$ a closed subgroup. We write $\mathsf{Cat}^G := \mathsf{Cat}_{\mathcal{S}^G} \simeq \mathsf{Fun}(\mathsf{Orb}_G^{op}, \mathsf{Cat}_\infty)$ for the ∞-category of G-categories. For a G-category $\underline{\mathcal{C}}$, we write its value at an orbit G/H by $\underline{\mathcal{C}}(G/H) := \mathcal{C}^H$.

> **Standard Facts about Orbit Category**
>
> For any morphism $\alpha : H \to G$ of compact Lie groups, there is an *induction functor*
>
> $$\mathsf{Ind}_\alpha^{\mathsf{Orb}} : \mathsf{Orb}^H \to \mathsf{Orb}^G$$
>
> sending an H-orbit S to $G \times_H S$, which will be a G-orbit. If $\alpha : H \to G$ is an epimorphism, we obtain a fully faithful right adjoint given by **restriction**:
>
> $$\mathsf{Res}_\alpha^{\mathsf{Orb}} : \mathsf{Orb}^G \to \mathsf{Orb}^H$$
>
> These can be constructed on the level of topologically enriched categories. For a closed subgroup $H \leq G$, induction induces an equivalence of topologically enriched categories
>
> $$\mathsf{Orb}^H \xrightarrow{\simeq} \mathsf{Orb}^G_{/(G/H)}$$
>
> whose inverse sends $T \to G/H$ to the homogeneous H-space given as the fiber over $eH \in G/H$, $T \times_{G/H} \{eH\}$.

For a continuous homomorphism $\alpha : K \to G$ of compact Lie groups we obtain two adjunctions



called *induction*, *restriction*, and *coinduction*, where Res_α is given by restriction along $\mathsf{Ind}_\alpha^{\mathsf{Orb}} : \mathsf{Orb}_H \to \mathsf{Orb}^G$, and $\mathsf{Ind}_\alpha = (\mathsf{Ind}_\alpha^{\mathsf{Orb}})_!$ while $\mathsf{CoInd}_\alpha = (\mathsf{Ind}_\alpha^{\mathsf{Orb}})_*$. When $\alpha = \theta : G \twoheadrightarrow G/N$ is an epimorphism (for $N \leq G$ closed normal subgroup), CoInd_θ admits a further right adjoint, which is written as CoInfl_θ (*coinflation*) given by right Kan extension along the fully-faithful right adjoint $\mathsf{Res}_\theta^{\mathsf{Orb}}$ to $\mathsf{Ind}_\theta^{\mathsf{Orb}}$. In particular, $\mathsf{Res}_\theta = (\mathsf{Ind}_\theta^{\mathsf{Orb}})^* \simeq (\mathsf{Res}_\theta^{\mathsf{Orb}})_!$ and $\mathsf{CoInfl}_\theta = (\mathsf{Res}_\theta^{\mathsf{Orb}})$ are fully-faithful in this case, and $\mathsf{CoInd}_\theta \simeq (\mathsf{Res}_\theta^{\mathsf{Orb}})^*$. Thus, we obtain the diagram of adjunctions:



Here the maps $N\backslash(-)$, Infl_θ, $(-)^{gN}$, and CoInfl_θ are called the **genuine quotient**, **inflation**, **genuine fixed points**, and **coinflation**, respectively. We often also write Infl_θ and CoInf_θ as Infl_G^Q and CoInfl_G^Q, respectively.

Note that for a G orbit G/H, its induction to G/N is given by the double coset

$$G/N \times^G G/H = N\backslash G/H \cong G/NH$$

since N is normal. From the left Kan extension formula defining the genuine quotient, we obtain

$$(N\backslash\underline{\mathcal{C}})((G/N)/K) \simeq \mathsf{coeq}\left(\coprod_{G/H \to G/H', (G/N)/K \to G/NH} \underline{\mathcal{C}}(G/H') \rightrightarrows \coprod_{G/H,(G/N)/.}\right.$$

In order to work with stability in the parameterized context, the naive definition of stably valued $\mathcal{B}$-categories is insufficient.

Fibrewise Pointed (resp. Stable) $\mathcal{B}$-categories

A $\mathcal{B}$-category $\underline{\mathcal{C}}$ is **fiberwise pointed** (resp. **fiberwise stable**) if the functor $\underline{\mathcal{C}} : \mathcal{B}^{op} \to \mathsf{Cat}_\infty$ factors through the subcategory $\mathsf{Cat}_{\infty,*}$ of pointed ∞-categories and pointed functors (resp. $\mathsf{Cat}_\infty{}^{\mathsf{st}} \subseteq \mathsf{Cat}_\infty$ the subcategory of stable ∞-categories and exact functors). We denote by $\mathsf{Cat}^{\mathsf{ptd}}_{\mathcal{B}}$ (resp. $\mathsf{Cat}^{\mathsf{st}}_{\mathcal{B}}$) the category of fiberwise pointed (resp. fiberwise stable) ∞-categories and pointed (resp. exact) functors.

A parameterized analogue of the category of spectra can be given for G-categories by the G-spectra $\underline{\mathsf{Sp}}^G : \mathsf{Orb}_G^{op} \to \mathsf{Cat}_\infty$ which at an orbit G/H is the ∞-category $\underline{\mathsf{Sp}}^G(G/H) := \mathsf{Sp}^H = \mathcal{S}^H_*[\{S^V\}^{-1}]$ of *genuine H-spectra*, where we formally invert the representation spheres of all finite dimensional G-representations. This is equivalent to the stabilization

$$\mathsf{Stab}_{\{S^V\}}(\mathcal{S}^H_*) = \operatorname*{colim}_{F \subseteq^{\mathsf{fin}} \{S^V\}} \mathsf{Stab}_{\otimes F}(\mathcal{S}^H_*)$$

where for a presentably symmetric monoidal ∞-category $\mathcal{C}$ and an object $x \in \mathcal{C}$,

$$\mathcal{C}[x^{-1}] \simeq \mathsf{Stab}_x(\mathcal{C}) = \mathsf{colim}\left(\mathcal{C} \xrightarrow{-\otimes x} \mathcal{C} \xrightarrow{-\otimes x} \mathcal{C} \xrightarrow{-\otimes x} \cdots\right)$$

Note that $\underline{\mathcal{S}}^G_* = \mathcal{S}^G_* \otimes_{\mathcal{S}^G} \Omega_{\mathcal{S}^G}$ and $\underline{\mathsf{Sp}}^G = \mathsf{Sp}^G \otimes_{\mathcal{S}^G} \Omega_{\mathcal{S}^G}$ as defined, where $- \otimes_{\mathcal{S}^G} \Omega_{\mathcal{S}^G} : \mathsf{Mod}_{\mathcal{S}^G}(\mathsf{Pr}^L) \to \mathsf{Pr}^L_G$ is as in[2]. The G-categories $\underline{\mathcal{S}}^G_*$ and $\underline{\mathsf{Sp}}^G$ are examples of idempotent algebras in Pr^L_G, so being modules over them is *property-like*.

When H is a finite group Sp^H coincides with the ∞-category of *spectral Mackey functors*

$$\mathsf{Fun}^{\oplus}(\mathsf{Span}(\mathsf{Fin}_H), \mathsf{Sp})$$

The G-category $\underline{\mathsf{Sp}}^G$ is not only fiberwise stable, it also satisfies a form of the **Wirthmuller isomorphism** in the sense that indexed products and coproducts over orbits G/H are canonically equivalent.

We will write $\mathsf{Pr}^{L,G_{\mathsf{st}}}_G \subseteq \mathsf{Pr}^{L,\mathsf{st}}_G \subseteq \mathsf{Pr}^L_G$ for the full subcategories on G-stable and fiberwise stable presentable G-categories, where a presentable G-category is G-

stable if and only if it is a module over $\underline{\mathsf{Sp}}^G$. For a finite group G, we can also define G-stable G-categories more generally, and so obtain a subcategory $\mathsf{Cat}_G^{G_{\mathrm{st}}} \subseteq \mathsf{Cat}^G$.

Universal Property of Presentable G-categories that are pointed and invert representation spheres

Suppose $\underline{\mathcal{D}}$ is a presentable G-category such that $\mathcal{D}^G$ is pointed and $- \otimes S^V : \mathcal{D}^G \to \mathcal{D}^G$ is an equivalence for any finite dimensionsal G-representation V. Then the restriction map

$$\mathsf{Fun}_G^L(\underline{\mathsf{Sp}}^G, \mathcal{D}) \to \mathsf{Fun}_G^L(\underline{\mathcal{S}}^G, \mathcal{D})$$

is an equivalence.

Proof.
Recall that $- \otimes \underline{\mathcal{S}}^G : \mathsf{Pr}^L \to \mathsf{Pr}_G^L$ is left adjoint to Γ. Then we obtain an equivalence

$$\mathsf{Fun}_G^L(\underline{\mathcal{S}}_*^G, \underline{\mathcal{D}}) \simeq \mathsf{Fun}^L(\mathcal{S}_*, \mathcal{D}^G) \xrightarrow{\simeq} \mathsf{Fun}^L(\mathcal{S}, \mathcal{D}^G) \simeq \mathsf{Fun}_G^L(\underline{\mathcal{S}}^G, \underline{\mathcal{D}})$$

using the $\mathcal{D}^G$ is pointed. Similarly, the restriction map

$$\mathsf{Fun}_G^L(\underline{\mathsf{Sp}}^G, \underline{\mathcal{D}}) \simeq \mathsf{Fun}_{\mathcal{S}_*^G}^L(\mathsf{Sp}^G, \mathcal{D}^G) \xrightarrow{\simeq} \mathsf{Fun}_{\mathcal{S}_*^G}^L(\mathcal{S}_G^*, \mathcal{D}^G) \simeq \mathsf{Fun}_G^L(\underline{\mathcal{S}}_*^G, \underline{\mathcal{D}})$$

is an equivalence due to the universal property of $\underline{\mathsf{Sp}}^G$. Thus, composing these equivalences we obtain the desired result.
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Characterization of (presentable) G-stability

For a presentable G-category $\underline{\mathcal{C}}$, the following are equivalent:

(1) $\underline{\mathcal{C}}$ is G-stable
(2) $\underline{\mathcal{C}}$ is fiberwise pointed and for all closed subgroups $H \leq G$ and all finite dimensional H-representations V, tensoring with $S^V \in \mathcal{S}_*^H$ induces an equivalence $- \otimes S^V : \mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^H$
(3) $\underline{\mathcal{C}}$ is fiberwise pointed and for all finite dimensional G-representations V, tensoring with $S^V \in \mathcal{S}_*^G$ induces an equivalence

$$- \otimes S^V : \underline{\mathcal{C}} \xrightarrow{\simeq} \underline{\mathcal{C}}$$

In many of the applications of interest we will need **isotropy separation** arguments. Thus, we'll recall some constructions on G-categories given a family $\mathcal{F}$ of subgroups of G. Recall that a *family of subgroups* of a compact Lie group G is a collection of *closed* subgroups of G which is closed under *subgroups* and *conjugation*.

Note: Conjugacy classes of subgroups of G correspond bijectively to isomorphism classes of objects in Orb^G. Give a collection S of closed subgroups of G that is closed under conjugacy, we set $\mathsf{Orb}^G_S \subseteq \mathsf{Orb}^G$ as the full subcategory on those G/H with $H \in S$. For example, we could take $S = \mathcal{F}^c$ the collection of all subgroups which lie in the compliment of a family $\mathcal{F}$. This never forms a family, except in the case of the empty family or the family of all subgroups.

A family for quotients

Suppose that $N \leq G$ is a closed normal subgroup of G. Let $\Gamma_N := \{H \leq G \mid N \not\leq H\}$, which is a family since N is a normal subgroup. Then Γ_N^c consists of those $H \leq G$ with $N \leq H$. Let $\alpha : G \to G/N$ be the quotient homomorphism. Observe that the adjunction $\mathsf{Ind}_\alpha^{\mathsf{Orb}} \dashv \mathsf{Res}_\alpha^{\mathsf{Orb}}$ restricts to an equivalence of categories

$$\mathsf{Ind}_\alpha^{\mathsf{Orb}} : \mathsf{Orb}^G_{\Gamma_N^c} \simeq \mathsf{Orb}^{G/N} : \mathsf{Res}_\alpha^{\mathsf{Orb}}$$

A family for free actions

Suppose that $N \leq G$ is a closed normal subgroup of G. Then we have a family $\mathcal{F}_N := \{H \subseteq G \mid H \cap N = \{1\}\}$, again using the fact that N is normal. Note that when $N \neq \{1\}$, we have an inclusion of families $\mathcal{F}_N \subseteq \Gamma_N$. Thus, there is an inclusion of collections $\Gamma_N^c \subseteq \mathcal{F}_N^c$.

Given a collection of subgroups closed under conjugacy S, we can define the category of S-categories as follows:

S-Categories

Let G be a compact Lie group and S a collection of subgroups closed under conjugacy. Then we write $\mathsf{Cat}_{G,S} := \mathsf{Fun}((\mathsf{Orb}^G_S)^{op}, \mathsf{Cat}_\infty)$ for the

category of S-categories.

If $\mathcal{F}$ is a family of closed subgroups of G, we now want to construct a variant of the *isotropy separation sequence* relating the categories Cat_G, $\mathsf{Cat}_{G,\mathcal{F}}$, and $\mathsf{Cat}_{G,\mathcal{F}^c}$. (c.f. Survey On Equivariant Stable Homotopy Theory - Scrap Notes).

Isotropy Separation and Coinduction

Consider a continuous epimorphism $\theta : G \twoheadrightarrow G/N$ of compact Lie groups. From the family for quotients example we have the functor $\mathsf{Res}_\theta^{\mathsf{Orb}} : \mathsf{Orb}^{G/N} \hookrightarrow \mathsf{Orb}^G$ which restricts to an equivalence $\mathsf{Orb}^{G/N} \simeq \mathsf{Orb}^G_{\Gamma_N^c}$. This identifies the adjunctions $\mathsf{Coind}_\theta : \mathsf{Cat}_G \leftrightarrows \mathsf{Cat}_{G/N} : \mathsf{Coinfl}_\theta$ and $j^* : \mathsf{Cat}_G \leftrightarrows \mathsf{Cat}_{G,\Gamma_N^c} : j_*$.

Singular Part

Consider the inclusion $j : \mathsf{Orb}^G_{\mathcal{F}^c} \hookrightarrow \mathsf{Orb}^G$. Then we get the *Bousfield colocalization* $j_! : \mathcal{S}^G_{\mathcal{F}^c} \leftrightarrows \mathcal{S}^G : j^*$ such that $j_! j^*(\underline{G/H}) = \underline{G/H}$ for every $H \notin \mathcal{F}$, and $j_! j^*(\underline{G/H}) = \emptyset$ for all $H \in \mathcal{F}$. Thus, $j_! j^*$ picks out the isotropy of a G-space $\underline{X}$ not in $\mathcal{F}$. We write

$$\underline{X}_{\mathcal{F}^c} := j^* \underline{X} \in \mathcal{S}^G_{\mathcal{F}^c} \quad \text{and} \quad \underline{X}_{\widetilde{\mathcal{F}}} := j_! j^* \underline{X} = j_! \underline{X}_{\mathcal{F}^c} \in \mathcal{S}^G$$

The adjunction counit $\epsilon : \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ then admits the interpretation as the **inclusion of the $\mathcal{F}$-singular part** of the G-space $\underline{X}$. It is the identity on G/H for $H \notin \mathcal{F}$, and the map $\emptyset \to \underline{X}(\underline{G/H})$ for $H \in \mathcal{F}$.

Example

For $\mathcal{F} = \mathcal{P}$ the family of proper subgroups, $\underline{X}_{\widetilde{\mathcal{P}}}$ is given by the fixed points space X^G, considered as a G-space with trivial action. For $\mathcal{F} = \{e\}$, the intuition for $\underline{X}_{\widetilde{\{e\}}}$ is that it gives the G-space of all points in $\underline{X}$ with non-trivial isotropy.

Now, fix a family $\mathcal{F}$ of closed subgroups of G. The adjunction $j^* : \mathsf{Cat}_G \leftrightarrows \mathsf{Cat}_{G,\mathcal{F}^c} : j_*$ does *not* in general restrict to an adjunction between presentable or (fiberwise) stable G-categories as the adjunction unit does not preserve G-colimits. However, we will see that the restriction

$j_* : \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c} \to \mathrm{Pr}^{L,\mathrm{st}}_{G}$, which is fully-faithful, does admit a different symmetric monoidal left adjoint $\tilde{j}^*$. This will come from the fact that j_* is actually a smashing localization.

> **(Co)tensoring over Pointed Groupoids**
>
> Let $\underline{\mathcal{E}}$ be a pointed $\mathcal{B}$-category admitting all parameterized (co)limits. Then $\underline{\mathcal{E}}$ is naturally tensored and cotensored over pointed $\mathcal{B}$-groupoids $\mathcal{B}_*$ as follows:
>
> For $\underline{*} \to \underline{X}$ in $\mathcal{B}_*$, and $E \in \underline{\mathcal{E}}$, we have
>
> $$\underline{X} \wedge E := \mathrm{cofib}\left(E \simeq \operatorname*{colim}_{\underline{*}} E \to \operatorname*{colim}_{\underline{X}} E\right)$$
>
> and
>
> $$\underline{\mathrm{hom}}_*(\underline{X}, E) := \mathrm{fib}\left(\lim_{\underline{X}} E \to \lim_{\underline{*}} E \simeq E\right)$$
>
> These exhibit $\underline{\mathcal{E}}$ as being tensored and cotensored over $\mathcal{B}_*$, respectively. For example, if $F \in \underline{\mathcal{E}}$ we have
>
> $$\begin{aligned} \mathrm{Map}_{\underline{\mathcal{E}}}(\underline{X} \wedge E, F) &\simeq \mathrm{fib}\left(\mathrm{Map}_{\underline{\mathcal{E}}}(\operatorname*{colim}_{\underline{X}} E, F) \to \mathrm{Map}_{\underline{\mathcal{E}}}(E, F)\right) \\ &\simeq \mathrm{Map}_{\mathcal{B}}(\underline{X}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E, F)) \times_{\mathrm{Map}_{\mathcal{B}}(\underline{*}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E,F))} \{*\} \\ &\simeq \mathrm{Map}_{\mathcal{B}_*}(\underline{X}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E, F)) \end{aligned}$$
>
> Note that these constructions give us adjunctions $\underline{X} \wedge - : \underline{\mathcal{E}} \leftrightarrows \underline{\mathcal{E}} : \underline{\mathrm{hom}}_*(\underline{X}, -)$. Moreover, this is associative.

We now move to defining *Brauer quotients* of G-categories with respect to a fixed family, which are a special case of **Verdier quotients** (here for a fully-faithful functor $\mathcal{C} \to \mathcal{T}$ of presentable stable ∞-categories, the **Verdier quotient** is the homotopy cofiber in $\mathrm{Pr}^{L,\mathrm{st}}$).

> **$\mathcal{F}$-Brauer Quotients**
>
> For a finite group G, we define the **$\mathcal{F}$-Brauer quotient** $\underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ of a small G-stable ∞-category $\underline{\mathcal{D}}$ as a G-stable ∞-category admitting a G-exact functor $\Phi^{\mathcal{F}} : \underline{\mathcal{D}} \to \underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ which, for all G-stable ∞-categories $\underline{\mathcal{E}}$ induces an equivalence

$$(\Phi^{\mathcal{F}})^* : \underline{\mathsf{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}/\langle\mathcal{F}\rangle, \underline{\mathcal{E}}) \xrightarrow{\simeq} \underline{\mathsf{Fun}}^{\mathrm{ex},\mathcal{F}=0}(\underline{\mathcal{D}}, \underline{\mathcal{E}})$$

where the ∞-category of the right is the full G-subcategory of $\underline{\mathsf{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathcal{E}})$ is the full G-subcategory of G-exact functors $F : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ such that $\mathsf{Res}^G_H F : \mathsf{Res}^G_H \underline{\mathcal{D}} \to \mathsf{Res}^G_H \underline{\mathcal{E}}$ is the zero functor for all $H \in \mathcal{F}$.

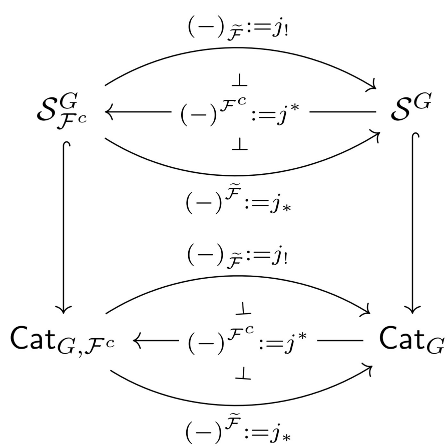
We write $\mathsf{Cat}^{G_{\mathrm{st}}}_{G,\mathcal{F}^c} \subseteq \mathsf{Cat}^{G_{\mathrm{st}}}_G$ for the full subcategory spanned by those G-stable ∞-categories lying in the image of j_* (i.e. those with value 0 on $\mathsf{Orb}^G_{\mathcal{F}}$).

Analogously, for G a general compact Lie group and $\mathcal{F}$ a family of closed subgroups, the $\mathcal{F}$-Brauer quotient of an object $\underline{\mathcal{C}} \in \mathsf{Pr}^{L,\mathrm{st}}_G$ as a presentable G-category $\underline{\mathcal{C}}/\langle\mathcal{F}\rangle$ equipped with a parameterized colimit-preserving functor $\Phi^{\mathcal{F}} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}/\langle\mathcal{F}\rangle$ inducing for every fiberwise stable presentable G-category $\underline{\mathcal{E}}$, an equivalence

$$(\Phi^{\mathcal{F}})^* : \underline{\mathsf{Fun}}^{L}(\underline{\mathcal{C}}/\langle\mathcal{F}\rangle, \underline{\mathcal{E}}) \xrightarrow{\simeq} \underline{\mathsf{Fun}}^{L,\mathcal{F}=0}(\underline{\mathcal{C}}, \underline{\mathcal{E}})$$

Fixed Point Notation

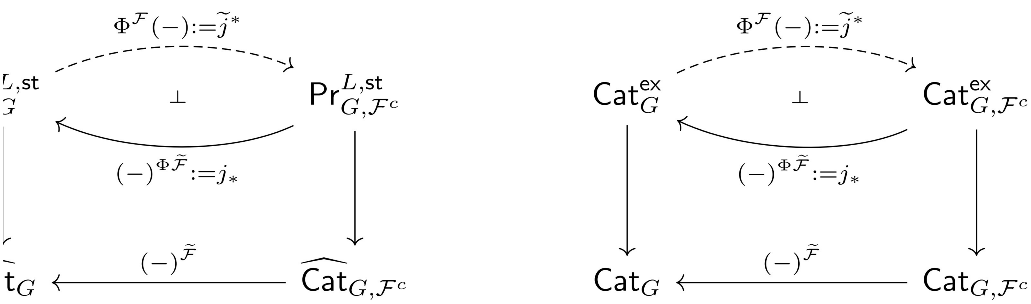
Let $j : \mathsf{Orb}^G_{\mathcal{F}^c} \hookrightarrow \mathsf{Orb}^G$ be the inclusion. Then we have a commuting diagram of adjunctions



Since $(-)_{\widetilde{\mathcal{F}}}$ and $(-)^{\widetilde{\mathcal{F}}}$ are fully-faithful, we also write $(-)_{\widetilde{\mathcal{F}}}$ and $(-)^{\widetilde{\mathcal{F}}}$ for $j_!j^*$ and j_*j^*, respectively. In particular, for $\underline{X} \in \mathcal{S}^G$, the counit gives a map $\epsilon : \underline{X}_{\widetilde{\mathcal{F}}} = j_!j^*\underline{X} \to \underline{X}$ as in the singular part. Additionally, isotropy separation implies that for $\underline{\mathcal{C}} \in \mathsf{Cat}_{G,\mathcal{F}^c}$ we have the description

$$\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}} \simeq \begin{cases} \underline{\mathcal{C}}(G/H) & \text{if } H \in \mathcal{F}^c; \\ * & \text{if } H \in \mathcal{F} \end{cases}$$

We also have the solid commuting squares

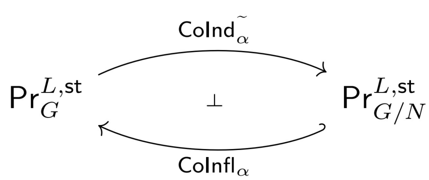


with the dashed maps left adjoints. The left diagram holds for general compact Lie groups G, whereas the right diagram is only defined for finite groups G from the existence of categorified Brauer quotients. As above, since $(-)^{\Phi\widetilde{\mathcal{F}}}$ is fully-faithful, we will also write $(-)^{\Phi\widetilde{\mathcal{F}}}$ for $j_*\widetilde{j}^*$. The adjunction unit will be denoted $\Phi^{\mathcal{F}} : (-) \to (-)^{\Phi\widetilde{\mathcal{F}}}$.

Now, let $N \leq G$ be a closed normal subgroup of the compact Lie group G, and write $\theta : G \twoheadrightarrow G/N$ for the quotient map. We now specialize the categorified Brauer quotient to the case of the family Γ_N to relate G- and G/N-stable ∞-categories.

Coinduction-Coinflation Adjunction (Prop 2.2.29[1-1])

Suppose that $\theta : G \twoheadrightarrow G/N$ is a continuous epimorphism of compact Lie groups. Then there is an adjunction



which is a smashing localization. The lax symmetric monoidal structure on CoInfl_{α} is equivalent to the lax symmetric monoidal structure from this smashing localization. We thus may view G/N-stable presentable ∞-categories as G-stable ∞-categories which vanish for all subgroups $H \leq G$ not containing N.

The symmetric monoidality of this adjunction then gives the following:

G/N-Spectra via Coinduction on G

Let $\theta : G \twoheadrightarrow G/N$ be the quotient map by a closed normal subgroup. Then the symmetric monoidal unit map $\underline{\mathsf{Sp}}^{G/N} \to \mathsf{CoInd}_{\alpha}^{\sim}\underline{\mathsf{Sp}}^{G}$ is an equivalence.

We can now define *geometric fixed points* in the parameterized context.

Geometric Fixed Points

Let $\theta_G : G \twoheadrightarrow 1$ be the quotient map. The symmetric monoidal G-colimit preserving unit map

$$\Phi^G : \underline{\mathsf{Sp}}^G \to \mathsf{CoInfl}_{\theta_G}\mathsf{CoInd}_{\theta_G}^{\sim}\underline{\mathsf{Sp}}^G \simeq \mathsf{CoInfl}_{\theta_G}\mathsf{Sp}$$

restricts to a symmetric monoidal colimit preserving functor $\Phi^G : \mathsf{Sp}^G \to \mathsf{Sp}$. There is an equivalence $\Phi^G \circ \Sigma^\infty_G(-) \simeq \Sigma^\infty(-)^G$ since Φ^G is $\mathcal{S}^G_*$-linear and sends the unit to the unit. These are precisely the properties that uniquely determine the classical geometric fixed points functor.

If $H \leq G$ is a closed subgroup, we have the symmetric monoidal G-colimit preserving functor $\Phi^H : \underline{\mathsf{Sp}}^G \to \mathsf{CoInd}^G_H\underline{\mathsf{Sp}}^H \to \mathsf{CoInd}^G_H\mathsf{CoInfl}_{\theta_H}\mathsf{Sp}$ which on global sections recovers the classical geometric fixed point functors $\Phi^H : \mathsf{Sp}^G \to \mathsf{Sp}$.

Geometric fixed points will be essential for reducing arguments about equivariant spectra to objects about ordinary spectra. This will rely on the fact that they geometric fixed points form a jointly conservative collection of functors.

Jointly Conservative Functors

A collection of $\mathcal{B}$-functors $\{F_s : \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ is **jointly conservative** if for all $X \in \mathcal{B}$, the collection $\{F_s(X) : \mathcal{C}(\underline{X}) \to \mathcal{D}_s(\underline{X})\}_{s\in S}$ is jointly conservative.

Note that if $\{F_s : \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ is a jointly conservative collection and $\underline{X} \in \mathcal{B}$, then the collection $\{F_s \circ - : \underline{\mathsf{Fun}}(\underline{X}, \underline{\mathcal{C}}) \to \underline{\mathsf{Fun}}(\underline{X}, \underline{\mathcal{D}}_s)\}_{s\in S}$ is also a jointly conservative collection.

Jointly Conservativity from Generating Objects

> If $\{\underline{X}_i\}_{i\in I}$ is a set of objects generating $\mathcal{B}$ under colimits, then a collection of $\mathcal{B}$-functors $\{F_s : \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ is jointly conservative if $\{F_s(X_i) : \underline{\mathcal{C}}(\underline{X}_i) \to \underline{\mathcal{D}}_s(\underline{X}_i)\}_{s\in S}$ is jointly conservative for each i.

The following result is the essential ingredient for the fixed point recognition principle discussed below.

> **Joint Conservativity of Geometric Fixed Points**
>
> The collection of G-functors
>
> $$\left\{\Phi^H : \underline{\mathsf{Sp}} \xrightarrow{\eta} \mathsf{CoInd}_H^G \mathsf{Res}_H^G \underline{\mathsf{Sp}} \xrightarrow{\mathsf{CoInd}_H^G \mathsf{Res}_H^G \Phi^{\mathcal{P}_H}} \mathsf{CoInd}_H^G \mathsf{Res}_H^G \underline{\mathsf{Sp}}^{\Phi \widetilde{\mathcal{P}}_H} \mid H \le G, \ \mathrm{c}$$
>
> is jointly conservative.

Equivariant Poincare Duality

We specialize the work in the previous talk Twisted Ambidexterity via Spivak Data to the case of the topos $\mathcal{S}^G = \mathsf{PSh}(\mathsf{Orb}^G)$ of G-spaces.

> **Spivak Datum in Equivariant Setting**
>
> Let $\underline{X} \in \mathcal{S}^G$ be a G-space and $\underline{\mathcal{C}}$ a symmetric monoidal G-category admitting $\underline{X}$-shaped colimits. A $\underline{\mathcal{C}}$-Spivak datum for $\underline{X}$ is a pair (ξ, c) where $\xi \in \mathsf{Fun}_G(\underline{X}, \underline{\mathcal{C}})$ is the **dualizing sheaf** and $c : \mathbb{1}_{\underline{\mathcal{C}}} \to X_! \xi$ is the **fundamental class**.
>
> Suppose $\underline{\mathcal{C}}$ admits $\underline{X}$-shaped limits and colimits and satisfies the $\underline{X}$-projection formula. For example, this holds if either
>
> **(a)** The G-category $\underline{\mathcal{C}}$ is presentably symmetric monoidal (i.e. $\in \mathsf{CAlg}(\mathsf{Pr}_G^L)$), or
> **(b)** if G is a finite group, $\underline{X} \in \mathcal{S}^G$ is compact, and $\underline{\mathcal{C}}$ were a small G-stably symmetric monoidal category (i.e. an object in $\mathsf{CAlg}(\mathsf{Cat}_G^{\mathsf{ex}})$)
>
> Under these conditions we can construct the *capping transformation* in $\underline{\mathsf{Fun}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}})$ (c.f. capping map)
>
> $$c \cap_\xi - : X_*(-) \to X_!(- \otimes \xi)$$

The spivak datum is **twisted ambidextrous** if the capping map is an equivalence, and it is additionally **Poincare** if ξ takes values in the subcategory $\underline{\mathsf{Pic}}(\underline{\mathcal{C}})$.

When $\underline{\mathcal{C}} \in \mathsf{CAlg}(\mathsf{Pr}^L_G)$ is a presentably symmetric monoidal G-category, from the presentable case for twisted ambidexterity we have that $\underline{X}$ is $\underline{\mathcal{C}}$-twisted ambidextrous if and only if it admits a twisted ambidextrous $\underline{\mathcal{C}}$-Spivak datum $(D_{\underline{X}}, \mathsf{Nm}^{-1}_{\underline{X}}(\mathbb{1}))$. Further, $\underline{X}$ is $\underline{\mathcal{C}}$-Poincare if additionally $D_{\underline{X}}$ takes values in $\underline{\mathsf{Pic}}(\underline{\mathcal{C}})$.

Unraveling $\underline{\mathsf{Sp}}$-Poincare Duality

Let $\underline{X} \in \mathcal{S}^G$ be a G-space. We want to consider local systems $\xi \in \mathsf{Fun}_G(\underline{X}, \underline{\mathsf{Sp}})$ that lands in $\underline{\mathsf{Pic}}(\underline{\mathsf{Sp}})$. Unraveling definitions this amounts to for each closed subgroup $H \leq G$ a local system of invertible H-spectra $\xi^H : X^H \to \mathsf{Pic}(\mathsf{Sp}^H)$, together with compatibilities that amount to providing for each map $\alpha : G/K \to G/H$, a homotopy in the diagram

$$\begin{array}{ccc} X^K & \xrightarrow{\xi^K} & \mathsf{Pic}(\mathsf{Sp}^K) \\ \uparrow{\scriptstyle \mathsf{Res}_\alpha} & \simeq & \uparrow{\scriptstyle \mathsf{Res}_\alpha} \\ X^H & \xrightarrow[\xi^H]{} & \mathsf{Pic}(\mathsf{Sp}^H) \end{array}$$

along with higher coherences between these homotopies. We also want a fundamental class $c : \mathbb{S}_G \to X_!\xi$ which amounts to an *equivariant homology class* of $\underline{X}$ with coefficients in the local system ξ. The capping map $c \cap_\xi - : X_*(-) \to X_!(- \otimes \xi)$ should then be thought of as the cap product with the homology class c.

Since $\underline{\mathsf{Sp}}$ is presentably symmetric monoidal, we can identify a large class of twisted ambidextrous objects using Theorem 4.8 in[2-1].

Compact G-Spaces are $\underline{\mathsf{Sp}}$-Twisted Ambidextrous

Every compact G-space $\underline{X}$ is $\underline{\mathsf{Sp}}$-twisted ambidextrous. Consequently, every compact G-space $\underline{X}$ is $\underline{\mathcal{C}}$-twisted ambidextrous for any G-stable presentably symmetric monoidal G-category.

The following result will be essential for constructing examples of G-Poincare spaces.

> **Fixed Point Recognition principle of Poincare Spaces**
>
> Suppose that $\underline{X} \in \mathcal{S}^G$ is a twisted ambidextrous G-space (e.g. a compact G-space) and let (ξ, c) be a $\underline{\mathsf{Sp}}^G$-Spivak datum for $\underline{X}$ such that $\xi : \underline{X} \to \underline{\mathsf{Sp}}^G$ takes values in $\underline{\mathsf{Pic}}(\underline{\mathsf{Sp}}^G)$. Then (ξ, c) exhibits $\underline{X}$ as a G-Poincare duality space if and only if for all closed subgroups $H \leq G$, the Spivak datum $\Phi^H(\xi, c)$ exhibits X^H as a non-equivariant Sp-Poincare space.

Examples

We now explore some sources of equivariant Poincare spaces. First, we start with the case where G is a compact Lie group. Recall that a **smooth G-manifold** is a smooth manifold on which G acts smoothly. An **equivariant embedding** of smooth G-manifolds is a smooth embedding between smooth G-manifolds that is also G-equivariant.

An **equivariant vector bundle** on a smooth G-manifold M is a tuple $\xi = (E, p)$ where E is a smooth G-manifold and $p : E \to M$ is an equivariant map which is also a vector bundle where G acts by bundle maps. For $x \in M^H$, the vector space $E_x := p^{-1}(x)$ carries an H-action by restriction.

> **Note:** Due to Illman's paper on *The equivariant triangulation theorem for actions of compact Lie groups*, smooth G-manifolds admit the structure of G-CW complexes which are necessarily finite for compact manifolds.

> **Facts from Equivariant Smooth Manifold Theory**
>
> **(i)** The tangent bundle of a smooth G-manifold can be naturally considered as an equivariant vector bundle (c.f. *p. 303 in Bredon's book*). If $f : M \to N$ is an equivariant embedding of smooth G-manifolds, then the equivariant tubular neighborhood theorem provides a smooth equivariant embedding $\nu(f) = f^*TN/TM$ into N (c.f. *Thm VI.2.2 in Bredon's book*)
> **(ii)** We write $\underline{M} \in \mathcal{S}^G$ for the underlying G-homotopy type of a smooth G-manifold M. Any G-vector bundle $p : E \to M$ over M defines a stable equivariant *spherical fibration* of the G-vector bundle $p : E \to M$. Furthermore, we can choose a *G-invariant Riemannian*

metric for p from which we obtain an associated unit disk bundle $D(p) \subseteq E$ and unit sphere bundle $S(p) \subseteq D(p)$. The fiberwise collapse maps $\underline{S^{E_x}} \to \mathsf{cofib}(\underline{S(p)_x} \to \underline{D(p)_x})$ for each $x \in M$ then assemble into a G-equivalence of G-spectra

$$M_!(\mathsf{Th}(p)) \xrightarrow{\simeq} \Sigma^\infty \mathsf{cofib}(\underline{S(p)} \to \underline{D(p)})$$

where $\mathsf{Th}(p)$ is the Thom-spectrum for the bundle viewed as a local system $\in \underline{\mathsf{Sp}}^{\underline{M}}$ given by fiberwise collapses.

(iii) For each G-manifold M, there exists an equivariant embedding into some G-representation V (c.f. **Mostow-Palais' Theorem**)

Closed Smooth G-manifolds give Poincare Spaces

Let M be a closed smooth G-manifold. Then the underlying G-space $\underline{M}$ is a G-Poincare space with dualizing object $\mathsf{Th}(TM)^{-1}$.

Proof.
Choose an embedding $f : M \hookrightarrow V$ into some G-representation. Denote the normal bundle of f by $\nu(f) = (p : E \to M)$ and pick a tubular neighborhood of M in V.

Consider the **Pontryagin-Thom collapse map**:

$$c : \mathbb{S} \xrightarrow{\simeq} \mathbb{S}^V \otimes \mathbb{S}^{-V} \to \Sigma^\infty \mathsf{cofib}\left(S^V \backslash (D(\nu) \backslash S(\nu)) \to S^V\right) \otimes \mathbb{S}^{-V}$$

$$\simeq \Sigma^\infty \mathsf{cofib}\left(S(\nu) \to D(\nu)\right) \otimes \mathbb{S}^{-V}$$

$$\simeq M_!(\mathsf{Th}(\nu) \otimes \mathbb{S}^{-V})$$

We want to show that the Spivak datum $(\mathsf{Th}(\nu) \otimes \mathbb{S}^{-V}, c)$ is Poincare. Since M is a compact G-space, $\mathsf{Th}(\nu)$ is invertible, so by the fixed point recognition principle it suffices to check that for every $H \subseteq G$, the Spivak datum $(\Phi^H \mathsf{Th}(\nu), \Phi^H c)$ is a Poincare Spivak datum for M^H. Recall that

$$\Phi^H \mathsf{Th}(\nu) : M^H \to \mathsf{Pic}(\mathsf{Sp}), \quad x \mapsto \Phi^H(\mathsf{Th}(\nu)(x)) = \Phi^H \Sigma^\infty S^{E_x} \simeq \Sigma^\infty S^{E_x^H}$$

But this is just the underlying *stable spherical fibration* of the normal bundle of M^H in V^H. The collapse map $\Phi^H c$ can be identified with the geometric Pontryagin-Thom collapse map of smooth manifold M^H embedded in V^H, so the Spivak datum is Poincare.

Now, note that the equivalence $\mathsf{const}_V = TV|_M \simeq \nu \oplus TM$ shows that

$$\mathsf{Th}(\nu) \otimes \mathbb{S}^{-V} \simeq \mathsf{Th}(\nu) \otimes \mathsf{Th}(\mathsf{const}_V)^{-1} \simeq \mathsf{Th}(\mathsf{const}_V) \otimes \mathsf{Th}(TM)^{-1} \otimes \mathsf{Th}(\mathsf{cons}$$

as desired.
□ ***

Another interesting source of equivariant Poincare duality spaces come from *generalized homotopy representations* due to Tom Dieck-Petrie.

Generalized Homotopy Representation

A **generalized homotopy representation** of a compact Lie group G is a compact G-space $\underline{V}$ such that for each closed subgroup $H \leq G$, the space V^H is equivalent to $S^{n(H)}$ for some $n(H) \in \mathbb{N}$. The function $n(-) : \mathsf{ClSubGrp}(G) \to \mathbb{N}$ is called the **dimension function** for the generalized homotopy representation.

Examples of generalized homotopy representations includes unit spheres of finite dimensional orthogonal G-representations, or one-point compactifications of finite dimensional linear G-representations. When the fixed points have CW-dimensions those of the respective spheres, then the generalized homotopy representation satisfies an *equivariant Hopf degree theorem*, i.e. G-homotopy classes of self maps are classified by their degree, an element in the *Burnside ring*.

Spivak data for Spheres

We construct a Spivak datum for $S^d \in \mathcal{S}$. Let $E := \mathsf{fib}(\Sigma^\infty_+ S^d \to \Sigma^\infty_+ * \simeq \mathbb{S})$. Then $E \simeq \mathbb{S}^d \in \mathsf{Pic}(\mathsf{Sp})$. Consider the composite

$$c : \mathbb{S} \xrightarrow{\simeq} E \otimes E^\vee \to \Sigma^\infty_+ S^d \otimes E^\vee \simeq S^d_!(S^d)^* E^\vee$$

We will see that $((S^d)^* E^\vee, c)$ is a Poincare Spivak datum for S^d. As S^d is stably parallelizable, its dualizing sheaf is constant with value $\mathbb{S}^{-d} \simeq E^\vee$. Now, assume $d \geq 1$. Now, $\pi_0 S^d_!(S^d)^* E^\vee \cong \mathbb{Z}$, and $c \in \pi_0 S^d_!(S^d)^* E^\vee \cong \mathbb{Z}$

> gives the collapse map of a Poincare Spivak datum if and only if it corresponds to a generator.

This observation motivates the following definition:

> **Homotopical Framing**
>
> A **homotopical framing** for $\xi \in \underline{\mathsf{Fun}}(\underline{X}, \underline{\mathsf{Sp}})$ is a G-spectrum E together with an equivalence $\xi \xrightarrow{\simeq} X^*E$. A compact G-space $\underline{X}$ is **homotopically parallelizable** if its dualizing sheaf $D_{\underline{X}} \in \underline{\mathsf{Fun}}(\underline{X}, \underline{\mathsf{Sp}})$ admits a homotopical framing.

> **Homotopy Framing for Generalized Homotopy Representations**
>
> The dualizing sheaf of a generalized homotopy representation $\underline{V}$ admits a canonical homotopical equivalence $D_{\underline{V}} \xrightarrow{\simeq} V^*\mathsf{fib}(\Sigma^\infty_+ V^\vee \to \Sigma^\infty_+ * \simeq \mathbb{S})^\vee$. In particular, generalized homotopy spheres are homotopically parallelizable G-Poincare spaces.

Proof.
As in the Spivak data for spheres, we have a map $c : \mathbb{S} \to E \otimes E^\vee \to \Sigma^\infty_+ V \otimes E^\vee \simeq V_! V^* E^\vee$, where we define

$$E^\vee := \mathsf{fib}\big(\Sigma^\infty_+ V^\vee \to \Sigma^\infty_+ * \simeq \mathbb{S}\big)^\vee$$

Upon taking geometric fixed points, the observation on the Spivak data for spheres identifies the composition

$$\Phi^H c : \Phi^H \mathbb{S} \to \Phi^H E \otimes \Phi^H E^\vee \to \Phi^H \Sigma^\infty_+ \underline{V} \otimes \Phi^H E^\vee \simeq V^H_! (V^H)^* \Phi^H E^\vee$$

as a Poincare Spivak datum for V^H. Then by the fixed point recognition principle we get that $(V^*E^\vee, c)$ is a Poincare G-Spivak datum for $\underline{V}$, and by the presentable case for twisted ambidexterity we get $D_{\underline{V}} \simeq V^*E^\vee$, as claimed.
□ ***

More generally, we can recognize whether a homotopically parallelizable space is G-Poincare by looking at its genuine fixed points for closed subgroups.

G-Poincare Space Detected at Fixed Points

Suppose $X \in (\mathcal{S}^G)^\omega$ is *homotopically parallelizable* and that X^H is a Poincare space for all closed subgroups $H \leq G$. Then X is a G-Poincare space.

Proof.
Since $\underline{X}$ is compact (as an object in the ∞-category $\mathcal{S}^G$), note that $X_! D_{\underline{X}} \simeq X_* X^* \mathbb{S}_G$ is a compact G-spectrum.

Now, suppose we have $E \in \mathsf{Sp}^G$ such that $D_{\underline{X}} \simeq X^* E$. As E is a retract of $X_! D_{\underline{X}} \simeq \Sigma^\infty_+ X \otimes E$, this implies that E is a compact G-spectrum. If all fixed points of X are Poincare spaces, then all geometric fixed points of E are invertible. Together this shows that E is invertible so that X is a G-Poincare space.
□ ***

References

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