Cardinalities and Semiadditive Height

Introduction/Motivation

These notes are for a 30~40 minute talk on cardinalities in semiadditive, along with the notion of semiadditive height, covering topics from sections 2.1-3.2 of the paper *Ambidexterity and Height*[1], which was given as part of an Ambidexterity seminar at UIUC in Fall 2025. For notation please refer to the previous note introducing semiadditivity.

Cardinalities

Recall from last time that if a map of π-finite spaces $q : A \to B$ is $\mathcal{C}$-ambidextrous for an ∞-category $\mathcal{C}$, then we obtain a *norm equivalence*

$$\mathsf{Nm}_q : q_! \overset{\simeq}{\Longrightarrow} q_*$$

where $q_! \dashv q^* \dashv q_*$, for $q^* : \mathcal{C}^B \to \mathcal{C}^A$ given by pre-composition. This norm map allows us to define integration of families of maps:

$$\int_q : \mathsf{Map}_{\mathcal{C}^A}(q^*X, q^*Y) \to \mathsf{Map}_{\mathcal{C}^B}(X, Y)$$

which can be given by the composite

$$\mathsf{Map}_{\mathcal{C}^A}(q^*X, q^*Y) \xrightarrow{q_!} \mathsf{Map}_{\mathcal{C}^B}(q_!q^*X, q_!q^*Y) \underset{\simeq}{\xleftarrow{-\circ\mathsf{Nm}_q}} \mathsf{Map}_{\mathcal{C}^B}(q_*q^*X, q_!q^*Y) \xrightarrow{\epsilon\circ-\circ\eta} \mathsf{Map}_{\mathcal{C}^B}(X, Y)$$

Integrating the identity morphism yields the notion of $\mathcal{C}$-cardinality.

> **$\mathcal{C}$-cardinality**
>
> Let $\mathcal{C} \in \mathsf{Cat}_\infty$ and let $A \xrightarrow{q} B$ be a $\mathcal{C}$-ambidextrous map. We have a natural transformation $\mathrm{id}_{\mathcal{C}^B} \xrightarrow{|q|_\mathcal{C}} \mathrm{id}_{\mathcal{C}^B}$ given by the composition
>
> $$\mathrm{id}_{\mathcal{C}^B} \xrightarrow{u_*} q_*q^* \underset{\simeq}{\xleftarrow{\mathsf{Nm}_q}} q_!q^* \xrightarrow{c_!} \mathrm{id}_{\mathcal{C}^B}$$
>
> For a $\mathcal{C}$-ambidextrous space A, we write $\mathrm{id}_\mathcal{C} \xrightarrow{|A|_\mathcal{C}} \mathrm{id}_\mathcal{C}$ and call $|A|_\mathcal{C}$ the **$\mathcal{C}$-cardinality** of A
> .

Note that for a given object $X \in \mathcal{C}$, $X \xrightarrow{|A|_X} X$ is exactly $\int_A \mathrm{id}_X$.

Motivating Example

Let $\mathcal{C}$ be a semiadditive ∞-category. For a finite set A, viewed as an 0-finite space, the operation $|A|_{\mathcal{C}}$ is simply the multiplication by the natural number which is the usual cardinality of A.

Note: For a $\mathcal{C}$-ambidextrous space A, the A-limits and A-colimits in $\mathcal{C}$ are canonically isomorphic, which implies the following:

Prop: Preservation of Limits and Colimits for Ambidextrous Spaces

Let $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}_{\infty}$, and let A be a $\mathcal{C}$- and $\mathcal{D}$-ambidextrous space. A functor $F : \mathcal{C} \to \mathcal{D}$ preserves all A-limits if and only if it preserves all A-colimits. Moreover, if F preserves all A-(co)limits, then $F(|A|_{\mathcal{C}}) \simeq |A|_{\mathcal{D}}$.

Using Fubini's theorem for iso-normed functors, we can obtain the following additivity result for cardinalities. In the current context Fubini's Theorem for iso-normed functors says that if $A \xrightarrow{p} B \xrightarrow{q} C$ are π-finite maps of π-finite spaces such that p and q are both $\mathcal{C}$-ambidextrous, then $\int_{qp}$ is homotopic to the composite

$$\mathsf{Map}_{\mathcal{C}^A}(p^*q^*X, p^*q^*Y) \xrightarrow{\int_p} \mathsf{Map}_{\mathcal{C}^B}(q^*X, q^*Y) \xrightarrow{\int_q} \mathsf{Map}_{\mathcal{C}^C}(X, Y)$$

Prop: Additivity of Cardinalities

Let $\mathcal{C} \in \mathsf{Cat}_{\infty}$ and $A \xrightarrow{q} B$ a map of spaces. If B and q are $\mathcal{C}$-ambidextrous, then A is $\mathcal{C}$-ambidextrous and for every $X \in \mathcal{C}$,

$$|A|_X = \int_B |q|_{B^*X}$$

Intuition: This says that the cardinality of the total space A is the *sum over B* of the cardinalities of the fibers A_b of q. To see how this is a consequence of Fubini we can re-write both sides using the integral notation to give

$$\int_A \mathrm{id}_{A^*X} \simeq \int_B \int_q \mathrm{id}_{q^*B^*X}$$

We can interpret this as saying

$$|A \times B|_{\mathcal{C}} = |A|_{\mathcal{C}} |B|_{\mathcal{C}} \in \mathsf{End}(\mathrm{id}_{\mathcal{C}})$$

and

$$|A|_{\mathcal{C}} = \coprod_{a \in \pi_0 A} |A_a|_{\mathcal{C}} \in \mathsf{End}(\mathrm{id}_{\mathcal{C}})$$

When $\mathcal{C}$ is monoidal and the tensor product preserves A-colimits in each variable, Lemma 3.3.4 of[2] implies that $|A|_X$ can be identified with $|A|_{\mathbb{1}} \otimes X$, where $\mathbb{1}$ is the monoidal unit. Additionally, if $R \in \mathsf{Alg}(\mathcal{C})$, then $|A|_R : R \to R$ can be identified with multiplication by the image of $|A|_{\mathbb{1}} \in \pi_0 \mathbb{1} := \pi_0 \mathsf{Map}(\mathbb{1}, \mathbb{1})$ under the unit map $\pi_0 \mathbb{1} \to \pi_0 R := \pi_0 \mathsf{Map}(\mathbb{1}, R)$, which we also denote by $|A|_R$.

Higher Commutative Monoids

We refer to an ∞-category as **m-semiadditive** if all m-finite spaces are ambidextrous. For $m = 0$ we recover the ordinary notion of a semiadditive ∞-category. Note that if $\mathcal{C} \subseteq \mathcal{D}$ is a full subcategory of an m-semiadditive ∞-category, then if $\mathcal{C}$ is either stable under m-finite colimits or m-finite limits, then it is stable under both, and it is m-semiadditive itself.

m-Commutative Monoids

Let $-2 \leq m < \infty$. For $\mathcal{C} \in \mathsf{Cat}_{\infty}^{m\mathsf{finLim}}$, the ∞-category of **m-commutative monoids** in $\mathcal{C}$ is given by

$$\mathsf{CMon}_m(\mathcal{C}) := \mathsf{Fun}^{m\mathsf{finR}}(\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})^{op}, \mathcal{C})$$

When $\mathcal{C} = \mathcal{S}$ we write $\mathsf{CMon}_m := \mathsf{CMon}_m(\mathcal{S})$, and refer to its objects as m-commutative monoids.

In the case $m = -2$, evaluating at pt, the unique object of $\mathsf{Span}(\mathcal{S}^{(-2)\mathsf{finColim}})$, gives an equivalence $\mathsf{CMon}_{-2}(\mathcal{C}) \simeq \mathcal{C}$.

Explication (CMon_m)

An object $X \in \mathsf{CMon}_m$ consists of an underlying space $X(\mathsf{pt})$, together with a collection of coherent operations for summation of m-finite families of points in it. Indeed, for $A \in \mathcal{S}_{m\mathsf{fin}}$, we have a canonical equivalence $X(A) \simeq X(\mathsf{pt})^A$. Given $A \to B$ in $\mathcal{S}_{m\mathsf{fin}}$, the image of $A = A \to B$ is the restriction $X(\mathsf{pt})^B \to X(\mathsf{pt})^A$, while the image of $B \leftarrow A = A$ encodes *integration along fibers* $X(\mathsf{pt})^A \to X(\mathsf{pt})^B$.

Question

How can we see the restriction and integration along fibers perspectives above?

Prop: Forgetful Functors between m-Commutative Monoids Cats

Let $-2 \leq m < \infty$ and let $\mathcal{C} \in \mathsf{Cat}_\infty^{(m+1)-\mathsf{finLim}}$. The restriction along the inclusion functor

$$\iota_m : \mathsf{Span}(\mathcal{S}_{m\mathsf{fin}}) \hookrightarrow \mathsf{Span}(\mathcal{S}_{(m+1)\mathsf{fin}})$$

induces a limit preserving functor

$$\iota_m^* : \mathsf{CMon}_{m+1}(\mathcal{C}) \to \mathsf{CMon}_m(\mathcal{C})$$

Proof.
It suffices to prove that ι_m preserves m-finite colimits. By the description of colimits in spans it suffices to prove that $\mathcal{S}_{m\mathsf{fin}} \hookrightarrow \mathcal{S}_{(m+1)\mathsf{fin}}$ is stable under m-finite colimits.
□

Question

How can we see that $\mathcal{S}_{m\mathsf{fin}}$ has m-finite colimits? If $A \xrightarrow{X} \mathcal{S}$ is an m-finite diagram, then

$$\mathsf{colim}_A X \simeq \mathsf{colim}_{A/X} * \simeq A/X$$

How do we know that A/X is also m-finite? We know that A is m-finite and that all fibers of the Kan fibration $A/X \to A$ are m-finite, so it is also an m-finite map. Do m-finite maps compose?

The following answers the above question:

Prop: m-finite Maps Compose

If $f : A \to B$ and $g : B \to C$ are m-finite, then so is their composite gf.

Proof.
Taking fibers, it suffices to show that if $f : A \to B$ is an m-finite map with B an m-finite space, then A is also m-finite. For each point $b \in B$, we have a homotopy fiber sequence $f^{-1}(b) \to A \to B$ where $f^{-1}(b)$ is also m-finite, by definition of m-finite maps. Thus, looking at the long exact sequence of homotopy groups for each $a \in f^{-1}(b)$, we see that A is also m-truncated, has finitely many path components, and has all homotopy groups begin finite, completing the proof.
□

We extend CMon_m to $m = \infty$ by defining for $\mathcal{C} \in \mathsf{Cat}_\infty^{\infty\mathsf{finLim}}$ the ∞-category

$$\mathsf{CMon}_\infty(\mathcal{C}) := \lim_m \mathsf{CMon}_m(\mathcal{C})$$

with limit computed in Cat_∞. This is equivalent to

$$\mathsf{Fun}^{\infty\mathsf{finR}}(\mathsf{Span}(\mathcal{S}_{\infty\mathsf{fin}})^{op}, \mathcal{C})$$

Consequently, when $\mathcal{C}$ is presentable, $\mathsf{CMon}_m(\mathcal{C})$ is presentable for all m, and $\mathsf{CMon}_\infty(\mathcal{C})$ can then be described as a colimit of $\mathsf{CMon}_m(\mathcal{C})$ in Pr^L:

Lemma: CMon_∞ as Colimit in Pr^L

For $\mathcal{C} \in \mathsf{Pr}^L$, the forgetful functors

$$\iota_m^* : \mathsf{CMon}_{m+1}(\mathcal{C}) \to \mathsf{CMon}_m(\mathcal{C})$$

admit left adjoints, and the colimit of the sequence

$$\mathcal{C} \simeq \mathsf{CMon}_{-2}(\mathcal{C}) \xrightarrow{\iota_{-1,!}} \mathsf{CMon}_{-1}(\mathcal{C}) \xrightarrow{\iota_{0,!}} \cdots$$

in Pr^L is $\mathsf{CMon}_\infty(\mathcal{C})$. In particular, $\mathsf{CMon}_\infty(\mathcal{C})$ is presentable.

The mapping spaces between two objects in an m-semiadditive ∞-category have a canonical m-commutative monoid structure.

Prop: Universality of $\mathsf{CMon}_m(-)$

Let $-2 \leq m \leq \infty$. For every $\mathcal{C} \in \mathsf{Cat}_\infty^{\oplus_m}$ and $\mathcal{D} \in \mathsf{Cat}_\infty^{m\mathsf{fin}}$, post-composition with evaluation at $\mathsf{pt} \in \mathcal{S}_{m\mathsf{fin}}$ induces an equivalence of ∞-categories

$$\mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}, \mathsf{CMon}_m(\mathcal{D})) \simeq \mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}, \mathcal{D})$$

As a consequence, for each m-semiadditive ∞-category we have a unique lift of the Yoneda embedding to a CMon_m-enriched Yoneda embedding:

Corollary: CMon_m-enriched Yoneda

Let $-2 \leq m \leq \infty$. For each $\mathcal{C} \in \mathsf{Cat}_\infty^{\oplus_m}$, there is a unique fully-faithful and m-semiadditive functor

$$よ^{\mathsf{CMon}_m} : \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{op}, \mathsf{CMon}_m)$$

whose composition with the forgetful functor $\mathsf{CMon}_m \to \mathcal{S}$ is the Yoneda embedding.

Here a functor between m-semiadditive ∞-categories is said to be **m-semiadditive** if it preserves m-finite limits.

Proof.

Taking $\mathcal{D} = \mathcal{S}$ in the universality of m-commutative monoids, we see that the ordinary Yoneda embedding

$$よ : \mathcal{C} \hookrightarrow \mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}^{op}, \mathcal{S}) \subseteq \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

lifts essentially uniquely to a fully-faithful m-finite limit preserving functor

$$よ^{\mathsf{CMon}_m} : \mathcal{C} \hookrightarrow \mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}^{op}, \mathsf{CMon}_m) \subseteq \mathsf{Fun}(\mathcal{C}^{op}, \mathsf{CMon}_m)$$

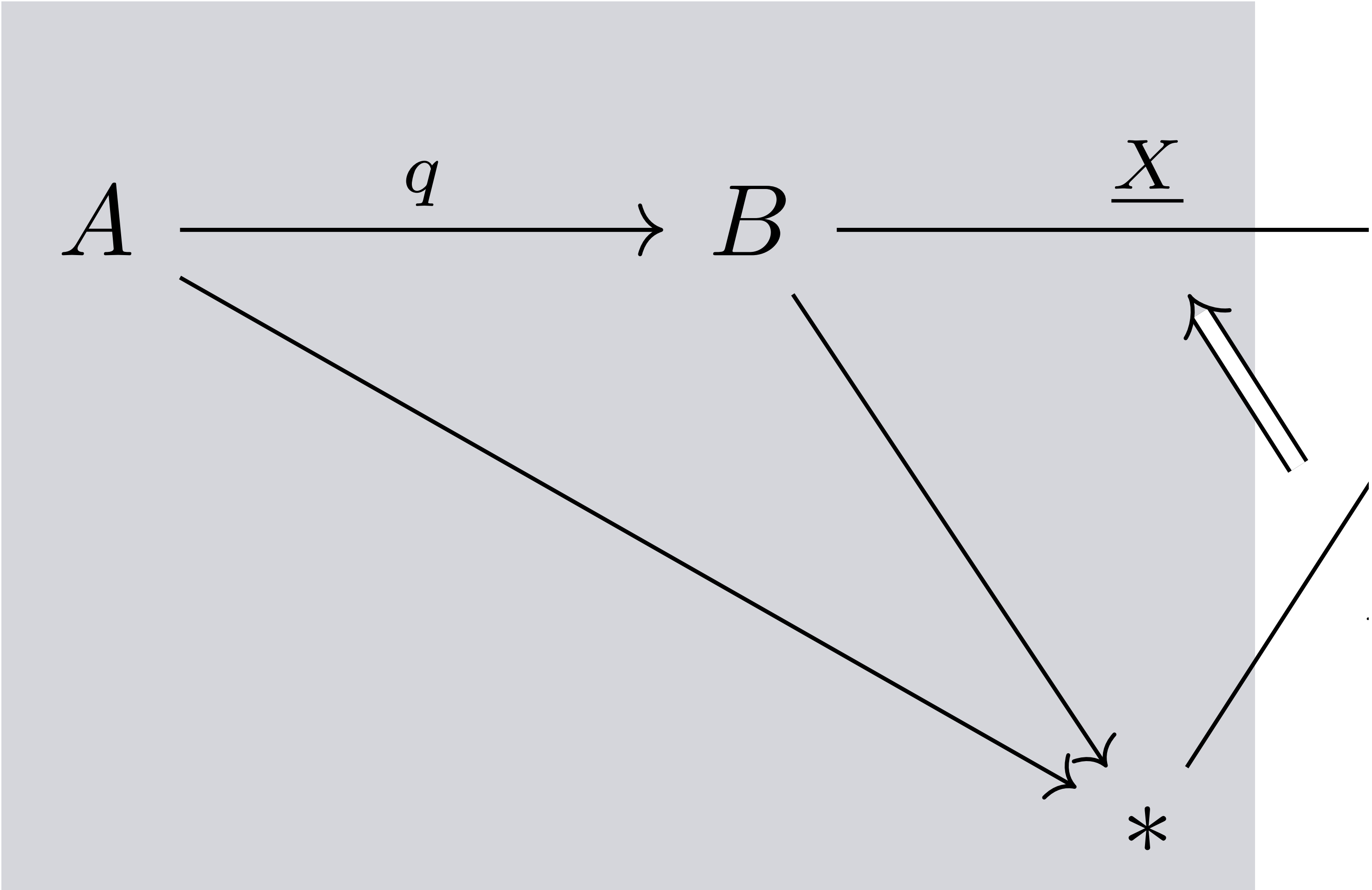
□

Currying we obtain a functor

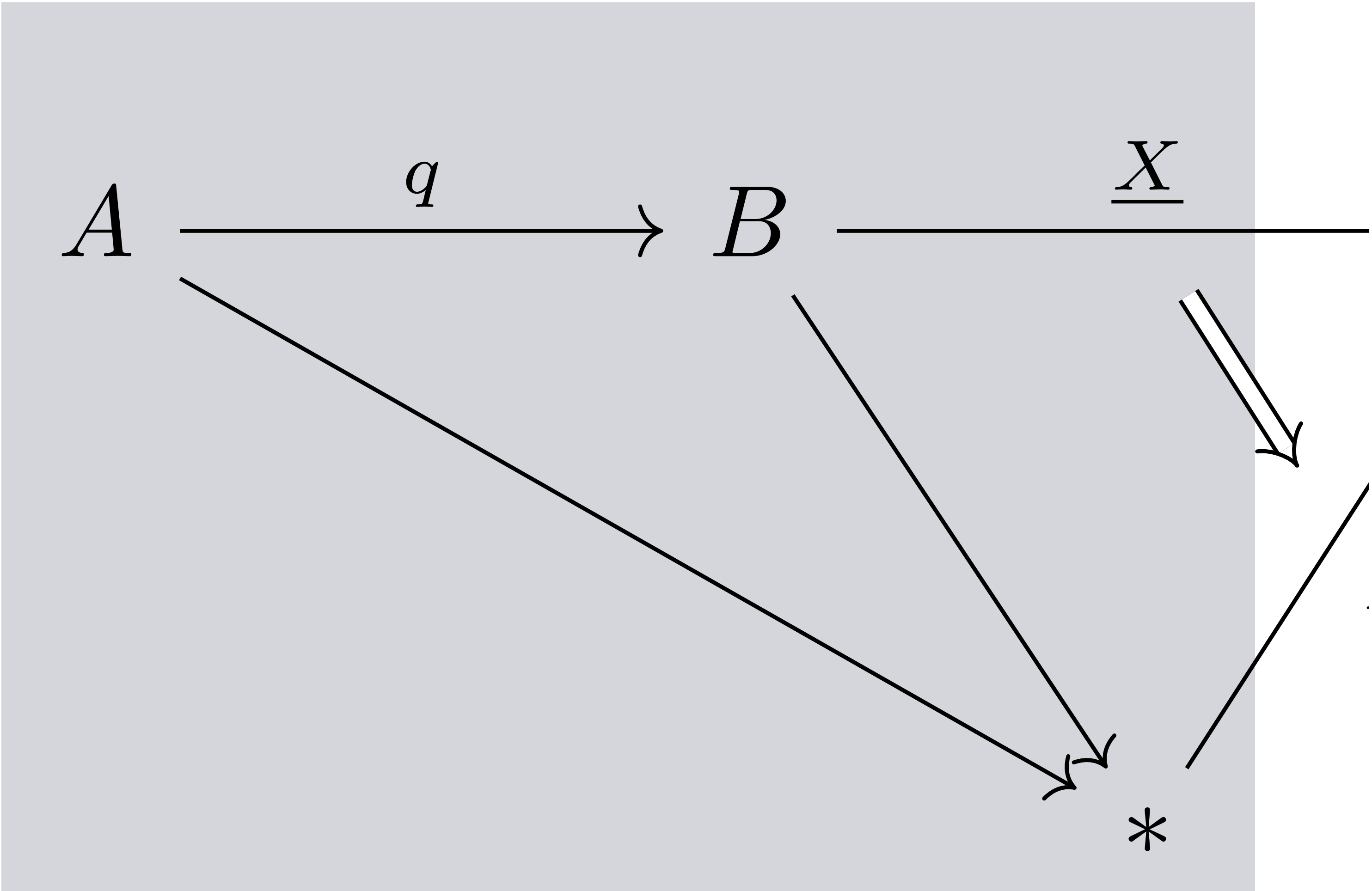
$$\mathsf{Hom}^{\mathsf{CMon}_m}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{CMon}_m$$

lifting $\mathsf{Map}_{\mathcal{C}}(-,-)$, and hence giving each mapping space a canonical m-commutative monoid structure.

Additionally, universality implies that if $\mathcal{C}$ is m-semiadditive, then the forgetful functor $\mathsf{CMon}_m(\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}$, given by evaluating at a point, is an equivalence. This implies that in an m-semiadditive ∞-category, then for every map of m-finite spaces $q : A \to B$, and every $X \in \mathcal{C}$, we have both a restriction map $q^* : X^B \to X^A$, induced the universal property of the limit, which can be expressed in terms of right Kan extensions as



as well as a transfer map $q_! : X^A \to X^B$ induced by the equivalences $\mathsf{Nm}_A : X[A] \xrightarrow{\simeq} X^A$ and $\mathsf{Nm}_B : X[B] \xrightarrow{\simeq} X^A$ and the universal property of the colimit, which can be expressed in terms of the left Kan extensions as



Examples

Before moving into more technical work, let's review some examples of m-semiadditive ∞-categories and the behaviour of cardinalities of m-finite spaces in them. We have the following universal example of an m-semiadditive ∞-category:

> **Universal Case**
>
> For $-2 \leq m < \infty$ the symmetric monoidal ∞-category of spans $\mathcal{C} = \mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})$ is the *universal m-semiadditive ∞-category*. For every $A \in \mathcal{S}_{m\mathsf{fin}}$, we have
>
> $$|A|_{\mathsf{pt}} = (\mathsf{pt} \leftarrow A \rightarrow \mathsf{pt}) \in \pi_0 \mathsf{Map}_{\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})}(\mathsf{pt}, \mathsf{pt})$$
>
> Note that $\pi_0 \mathsf{Map}_{\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})}(\mathsf{pt}, \mathsf{pt})$ is the set of isomorphism classes of m-finite spaces with semiring structure given by
>
> $$|A| + |B| = |A \sqcup B|, \;\; |A| \cdot |B| = |A \times B|$$
>
> Similarly, CMon_m is the universal *presentable m-semiadditive ∞-category*. The Yoneda embedding induces a fully-faithful m-semiadditive symmetric monoidal functor
>
> $$\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}}) \hookrightarrow \mathsf{CMon}_m$$

taking an m-finite space A to the free m-commutative monoid on A.

Homotopy Cardinality

For a π-finite space A, the **homotopy cardinality** of A is the rational number

$$|A|_0 := \sum_{a\in\pi_0(A)} \prod_{n\geq 1} |\pi_n(A,a)|^{(-1)^n} \in \mathbb{Q}_{\geq 0}$$

We say an ∞-category $\mathcal{C}$ is **semirational** if it is 0-semiadditive (i.e. 0-finite spaces are $\mathcal{C}$-ambidextrous, which are contractible, empty, and discrete spaces) and for each $n \in \mathbb{N}$, multiplication by n is invertible in $\mathcal{C}$ (e.g. $\mathsf{Sp}_{\mathbb{Q}}$ or $\mathbb{Q}\mathsf{Mod}$). Here multiplication by n on an object C is given by the cardinality $|\mathsf{pt}^{\sqcup n}|_C$, which is the composite

$$C \xrightarrow{\Delta} C^{\times n} \xleftarrow[\cong]{\mathsf{Nm}_{\mathsf{pt}^{\sqcup n}}} C^{\sqcup n} \xrightarrow{\nabla} C$$

A semirational ∞-category which admits all 1-finite colimits is automatically ∞-semiadditive, and for every π-finite space A, we have that its cardinality is its homotopy cardinality:

$$|A|_{\mathcal{C}} = |A|_0 \in \mathbb{Q}_{\geq 0} \subseteq \mathsf{End}(\mathrm{id}_{\mathcal{C}})$$

This comes from the fact that the cardinality is additive, and for every fiber sequence of π-finite spaces $F \to A \to B$ where B is connected, $|A| = |F||B|$.

In Chromatic homotopy theory we often come across examples of ∞-semiadditive ∞-categories of higher height. For a given prime p, and $0 \leq n < \infty$, let $K(n)$ be the Morava K-theory spectrum of height n at the prime p. We have that the localizations $\mathsf{Sp}_{K(n)}$ and $\mathsf{Sp}_{T(n)}$ are ∞-semiadditive. For $n = 0$, $\mathsf{Sp}_{K(0)} \simeq \mathsf{Sp}_{T(0)} \simeq \mathsf{Sp}_{\mathbb{Q}}$, and the cardinalities recover the homotopy cardinality. Similarly, since $\mathsf{Sp}_{K(n)}$ is p-local for all n, if A is a π-finite space whose homotopy groups have cardinality prime to p, then the $K(n)$-local cardinality of A coincides with the homotopy cardinality for all n by the previous example. However, this does not hold in general for π-finite spaces whose cardinality is not prime to p.

To study the $K(n)$-local cardinalities of π-finite spaces, it is useful to consider their image in Morava E-theory. For $n \geq 1$, let E_n be the Morava E-theory associated with some formal group of height n over $\overline{\mathbb{F}}_p$, viewed as an object of $\mathsf{CAlg}(\mathsf{Sp}_{K(n)})$. In particular, we have a (non-canonical) isomorphism

$$\pi_* E_n \cong \mathbb{W}(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u_i| = 0, \quad |u| = 2$$

Chromatic Cardinality

The ∞-category $\Theta_n := \mathsf{Mod}_{E_n}(\mathsf{Sp}_{K(n)})$ is ∞-semiadditive by Theorem 5.3.1 in6, and hence we can consider cardinalities of π-finite spaces in $\pi_0 E_n$. The **p-typical height n cardinality** of a π-finite space A is defined to be

$$|A|_n := |A|_{\Theta_n} \in \pi_0 E_n$$

For $n = 0$ we can identify $\overline{\mathbb{Q}}$ with $\pi_0 E_0$, and so can recover the homotopy cardinality. For $n > 0$, let $\widehat{L}_p A := \mathsf{Map}(B\mathbb{Z}_p, A)$ be the p-adic free loop space of A. It turns out that $|A|_n \in \pi_0 E_n$ belongs to the subring $\mathbb{Z}_{(p)} \subseteq \pi_0 E_n$ and satisfies $|A|_n = |\widehat{L}_p A|_{n-1}$. Applying this inductively we see that

$$|A|_n = |\mathsf{Map}(B\mathbb{Z}_p^n, A)|_0 \in \mathbb{Z}_{(p)}$$

for the p-typical height n cardinality in terms of the homotopy cardinality. If A is a p-space, then $\widehat{L}_p A \simeq LA := \mathsf{Map}(S^1, A)$ coincides with the ordinary loop space.

Question

How can we show that $\widehat{L}_p A \simeq LA$ when A is a p-space? Hint: First consider the universal examples $K(\mathbb{Z}/p, n)$.

The following gives another family of examples of higher semiadditive ∞-categories:

Prop: $\mathsf{Cat}_\infty^{m\text{finColim}}$ is m-semiadditive

For every $-2 \leq m \leq \infty$ the ∞-category $\mathsf{Cat}_\infty^{m\text{finColim}}$ is m-semiadditive.

Categorical Cardinality

Let $-2 \leq m \leq \infty$ and let $\mathcal{C} \in \mathsf{Cat}_\infty^{m\text{finColim}}$. For every m-finite space A, the m-semiadditive structure of $\mathsf{Cat}_\infty^{m\text{finColim}}$ gives rise to a functor $|A|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. When $m < \infty$, $|A|_{\mathcal{C}} \simeq \mathsf{colim}_A \Delta_{(-)}$ is given by taking the constant colimit on A. Since $\mathsf{Cat}_\infty^{\infty\text{finColim}} \to \mathsf{Cat}_\infty^{m\text{finColim}}$ preserves limits, and hence is m-semiadditive, the same claim holds for $m = \infty$.

Conversely, the m-semiadditive structure on $\mathsf{Cat}_\infty^{m\text{finLim}}$ is given by taking limits of constant diagrams.

(co)Cartesian m-commutative Monoid Structure

For $\mathcal{C} \in \mathrm{Cat}_\infty^{m\mathrm{finColim}}$, since $\mathcal{S}_{m\mathrm{fin}}$ is freely generated from a point under m-finite colimits, we have

$$\mathrm{Map}^{\mathrm{mfinL}}(\mathcal{S}_{m\mathrm{fin}}, \mathcal{C}) \simeq \mathrm{Map}(\mathrm{pt}, \mathcal{C}) \simeq \mathcal{C}^{\simeq}$$

and the resulting m-commutative monoid structure on $\mathcal{C}^{\simeq}$ is referred to as the **cocartesian structure**. Dually, for $\mathcal{C} \in \mathrm{Cat}_\infty^{m\mathrm{finLim}}$, we have

$$\mathrm{Map}^{m\mathrm{finR}}(\mathcal{S}_{m\mathrm{fin}}^{op}, \mathcal{C}) \simeq \mathrm{Map}(\mathrm{pt}, \mathcal{C}) \simeq \mathcal{C}^{\simeq}$$

and the resulting m-commutative monoid structure on $\mathcal{C}^{\simeq}$ is referred to as the **cartesian structure**.

The full subcategory $\mathrm{Cat}_\infty^{\oplus_m} \subseteq \mathrm{Cat}_\infty^{m\mathrm{finColim}}, \mathrm{Cat}_\infty^{m\mathrm{finLim}}$ is closed under colimits, and in particular is m-semiadditive, since the inclusion admits the right adjoint $\mathrm{CMon}_m(\mathcal{D})$.

Height

We will explore the notion of *semiadditive height*, as well as its relation to other classical notions of height. The definition of height will depend on a choice of a prime $p \in \mathbb{Z}$, and a *p-typical* version of m-semiadditivity where we use m-finite p-spaces rather than all m-finite spaces.

p-Spaces

Recall that a space X is a p-space if and only if all its homotopy groups are p^∞-torsion (i.e. for each $n \geq 1$, and each $x \in \pi_n X$, there exists $k \geq 1$ such that $p^k x = 0$). When X has finite homotopy groups, this implies that they are p-groups (i.e. their order is a power of p).

p-Typical Semiadditivity

Let p be a prime and $0 \leq m \leq \infty$. We say that

1. An ∞-category $\mathcal{C}$ is **p-typically m-semiadditive** if all m-finite p-spaces are $\mathcal{C}$-ambidextrous.
2. A functor $F : \mathcal{C} \to \mathcal{D}$ between such is **p-typically m-semadditive** if it preserves all m-finite p-space colimits (or equivalently limits).
3. An $\mathcal{O}$-monoidal ∞-category $\mathcal{C}$ for an ∞-operad $\mathcal{O}$ is **p-typically m-semiadditively $\mathcal{O}$-monoidal** if it is p-typically m-semiadditive and is compatible with m-finite p-space colimits (equivalently limits).

Let $\mathsf{Cat}_\infty^{\oplus_{m,p}} \subseteq \mathsf{Cat}_\infty$ denote the sub-∞-category of p-typically m-semiadditive ∞-categories and p-typically m-semiadditive functors.

> **Prop: p-Typical m-Semiadditivity from Building Blocks**
>
> Let $0 \leq m \leq \infty$.
>
> - **(1)** An ∞-category $\mathcal{C} \in \mathsf{Cat}_\infty{}^{\oplus_0}$ is p-typically m-semiadditive if and only if $B^k C_p$ is $\mathcal{C}$-ambidextrous for all $k = 1, \ldots, m$
> - **(2)** For $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}_\infty^{\oplus_{m,p}}$, a 0-semiadditive functor $F : \mathcal{C} \to \mathcal{D}$ is p-typically m-semiadditive if and only if it preserves $B^k C_p$-(co)limits for all $k = 1, \ldots, m$.

Proof.
(1) Since $B^k C_p = K(\mathbb{Z}/p, k)$ is an m-finite p-space for all $1 \leq k \leq m$, the only if direction is definitional. Conversely, let A be an m-finite p-space. Since $\mathcal{C}$ is 0-semiadditive, we are reduced to the case that A is connected. Indeed, otherwise $A \simeq \coprod_{i=1}^{N} A_i$ for A_i connected m-finite p-spaces. Then

$$\mathcal{C}^A \simeq \prod_{i=1}^{N} \mathcal{C}^{A_i}$$

and $\mathsf{colim}_A, \mathsf{lim}_A : \mathcal{C}^A \to \mathcal{C}$ can be given by $\prod_{i=1}^{N} \mathsf{colim}_{A_i}$ and $\prod_{i=1}^{N} \mathsf{lim}_{A_i}$ by iterating Kan extensions, so as $\mathcal{C}$ is 0-semiadditive it suffices that the norm maps $\mathsf{colim}_{A_i} \to \mathsf{lim}_{A_i}$ are equivalences.

Now, since A is connected, the Postnikov tower of A can be refined to a tower of principal fibrations

$$A \simeq A_r \to \cdots \to A_1 \to A_0 \simeq \mathsf{pt}$$

such that the fiber of each $A_i \to A_{i-1}$ is of the form $B^{k_i} C_p$ for some $1 \leq k_i \leq m$, since all connected π-finite p-spaces are C_p-nilpotent.

Since we can iterate Kan extensions, to show A is $\mathcal{C}$-ambidextrous it suffices to show each $A_i \to A_{i-1}$ is $\mathcal{C}$-ambidextrous, and since $\mathcal{C}$-ambidexterity is a fiber-wise condition, this follows from the fact that $B^{k_i} C_p$ is $\mathcal{C}$-ambidextrous.

(2) Analogously to **(1)**, we can reduce to connected m-finite p-spaces, at which point we can take a tower of principal fibrations, so that commuting with A-(co)limits follows from commuting with $(A_i \to A_{i-1})$-(co)limits, which are equivalent to commuting with fiber-wise indexed (co)limits, i.e. $B^{k_i} C_p$-(co)limits.
□

For a p-typical m-semiadditive ∞-category $\mathcal{C}$, the cardinalities $|B^n C_p|$ for $0 \leq n \leq m$ will play an important role in our definition of semiadditive height. The motivating example to consider in what follows is the following:

E_n-modules of $K(n)$-local Spectra

For $\mathcal{C} = \mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$, we have

$$|B^k C_p|_n = p^{\binom{n-1}{k}}$$

for all $n, k \geq 0$, where the $n = 0$ case is interpreted using $\binom{-1}{k} = (-1)^k$.

We now move to defining semiadditive height. This relies on the following notion of divisibility and completeness with respect to natural endomorphisms of the identity.

Divisibility and Completeness

Let $\mathcal{C} \in \mathrm{Cat}_\infty$ and let $\alpha : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$ be a natural endomorphism. An object $X \in \mathcal{C}$ is called:

1. **α-divisible** if α_X is invertible
2. **α-complete** if $\mathrm{Map}(Z, X) \simeq \mathrm{pt}$ for all α-divisible Z

We suggestively write $\mathcal{C}[\alpha^{-1}], \widehat{\mathcal{C}}_\alpha \subseteq \mathcal{C}$ for the full subcategories spanned by the α-divisible and α-complete objects, respectively.

Semiadditive Height of Objects

Let $\mathcal{C}$ be a p-typical m-semiadditive ∞-category and let $0 \leq n \leq m < \infty$. We define the **p-typical semiadditive height** of X as follows for $X \in \mathcal{C}$:

- **(1)** We say $\mathrm{ht}_p(X) \leq n$ if X is $|B^n C_p|$-divisible
- **(2)** We say $\mathrm{ht}_p(X) > n$ if X is $|B^n C_p|$-complete
- **(3)** We say $\mathrm{ht}_p(X) = n$ if $\mathrm{ht}_p(X) \leq n$ and $\mathrm{ht}_p(X) > n - 1$.

Warning

When $\mathcal{C}$ is not ∞-semiadditive, the notion of semiadditive height is *not* well-defined for all objects in $\mathcal{C}$. Indeed, we can only *test* an objects height being $n \geq$ or $n <$ if n is $\leq$ to the semiadditivity of $\mathcal{C}$, so we can have objects which have height $m <$, and not have a finite defined height.

In particular, the statements $\mathsf{ht}_p(X) \le n$ and $\mathsf{ht}_p(X) > n$ signify a certain **property** that X satisfies, and $\mathsf{ht}_p(X)$ is in general *not* a well-defined number which can be compared with n.

For $X \in \mathcal{C}$ which is ∞-semiadditive we write $\mathsf{ht}_p(X) = \infty$ if and only if $\mathsf{ht}_p(X) > k$ for all $k \ge 0$. By convention $-1 < \mathsf{ht}_p(X) \le \infty$ for all $X \not\simeq 0$, and $\mathsf{ht}_p(X) \le -1$ or $\mathsf{ht}_p(X) > \infty$ if and only if $X \simeq 0$.

Height 0

Let $\mathcal{C}$ be a 0-semiadditive ∞-category. Then an object $X \in \mathcal{C}$ is of height 0 if and only if $X \xrightarrow{p\cdot} X$, the map obtained by

$$X \xrightarrow{\Delta} \prod_{i=1}^{p} X \xleftarrow{\simeq} \coprod_{i=1}^{p} X \xrightarrow{\nabla} X$$

is an equivalence, and of height $\mathsf{ht}_p(X) > 0$ if it is p-complete.

The next result helps justify the inequality notation.

Prop: Inequalities of Height

Let $\mathcal{C}$ be a p-typical m-semiadditive ∞-category and let $0 \le n_0 \le n_1 \le m$ be some integers. Then for $X \in \mathcal{C}$

- **(1)** If $\mathsf{ht}_p(X) \le n_0$ then $\mathsf{ht}_p(X) \le n_1$
- **(2)** If $\mathsf{ht}_p(X) > n_1$ then $\mathsf{ht}_p(X) > n_0$

Proof.
For **(1)** it suffices by iterating that if $\mathsf{ht}_p(X) \le n$ for some $n \le m-1$, then $\mathsf{ht}_p(X) \le n+1$. Consider the principal fiber sequence

$$B^n C_p \to \mathsf{pt} \to B^{n+1} C_p$$

By assumption all maps and spaces in this sequence are $\mathcal{C}$-ambidextrous. Since $\mathsf{ht}_p(X) \le n$, we have that $|B^n C_p|_X$ is invertible. By the cardinality decomposition for principal fibrations we get

$$|B^{n+1} C_p|_X |B^n C_p|_X = |\mathsf{pt}|_X = \mathrm{id}_X$$

Thus, since $|B^n C_p|_X$ is invertible, so is $|B^{n+1} C_p|_X$, and in fact it is its inverse, and hence $\mathsf{ht}_p(X) \le n+1$.

(2) now follows since **(1)** showed that $|B^{n_0}C_p|$-divisible spaces are also $|B^{n_1}C_p|$-divisible.
□

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p-Typical Height in Stable ∞-Categories

If $\mathcal{C}$ is a stable ∞-category with non-trivial object X, and if $\mathbf{ht}_p(X) > 0$, then for any other prime ℓ we cannot have $\mathbf{ht}_\ell(X) > 0$. Indeed, if $\ell \neq p$ is another prime such that $\mathbf{ht}_\ell(X) > 0$, then this says that for all **TBC**

For $\mathcal{C}$ a p-typical m-semiadditive ∞-category we define

$$\mathcal{C}_{\leq n} := \mathcal{C}[|B^n C_p|^{-1}], \quad \mathcal{C}_{>n} = \widehat{\mathcal{C}}_{|B^n C_p|}, \quad \mathcal{C}_n = \mathcal{C}_{\leq n} \cap \mathcal{C}_{>n-1}$$

where $\mathcal{C}_n = \mathcal{C}[|\widehat{B^n C_p}|^{-1}]_{|B^{n-1}C_p|} = \widehat{\mathcal{C}}_{|B^{n-1}C_p|}[|B^n C_p|^{-1}]$.

Height of a p-typical m-semiadditive ∞-category

If $\mathcal{C}$ is a p-typical m-semiadditive ∞-category and $0 \leq n \leq m \leq \infty$, then we write

- **(1)** If $\mathcal{C} = \mathcal{C}_{\leq n}$, then $\mathbf{Ht}_p(\mathcal{C}) \leq n$
- **(2)** If $\mathcal{C} = \mathcal{C}_{>n}$, then $\mathbf{Ht}_p(\mathcal{C}) > n$
- **(3)** If $\mathcal{C} = \mathcal{C}_n$, then $\mathbf{Ht}_p(\mathcal{C}) = n$.

These constructions all form p-typical m-semiadditive ∞-categories.

Prop: p-typical m-semiadditive ∞-categories from Height Filtration

Let $\mathcal{C}$ be a p-typical m-semiadditive ∞-category and let $0 \leq n \leq m$. Then the subcategories $\mathcal{C}_{\leq n}, \mathcal{C}_{>n}, \mathcal{C}_n$ are stable under limits in $\mathcal{C}$. In particular, they are all p-typically m-semiadditive, and are furthermore m-semiadditive if $\mathcal{C}$ is.

This holds in fact for $\widehat{\mathcal{C}}_\alpha$ and $\mathcal{C}[\alpha^{-1}]$, with $\alpha : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$ an arbitrary natural endomorphism.

Prop: Height Can Only Decrease

If $F : \mathcal{C} \to \mathcal{D}$ is a map in $\mathbf{Cat}_\infty^{\oplus_{m,p}}$, then for all $X \in \mathcal{C}$ and $0 \leq n \leq m$, if $\mathbf{ht}_{\mathcal{C},p}(X) \leq n$ then $\mathbf{ht}_{\mathcal{D},p}(F(X)) \leq n$. If F is conservative, then the converse holds as well.

This is immediate from the fact that functors preserve equivalences and F maps $|B^nC_p| : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$ to $|B^nC_p| : \mathrm{id}_\mathcal{D} \Rightarrow \mathrm{id}_\mathcal{D}$. The following shows that the statement for the opposite inequalities does *not* in general hold.

Example

The 0-semiadditive functor $L_\mathbb{Q} : \mathsf{Sp}_{(p)} \to \mathsf{Sp}_\mathbb{Q}$ maps the p-complete sphere $\widehat{\mathbb{S}}_p$ which is of height > 0 to a non-zero object $\mathbb{Q} \otimes \widehat{\mathbb{S}}_p$ of height 0.

For inclusions however we do get such a result.

Prop: Height w.r.t Inclusions

Let $\mathcal{C}$ be a p-typical m-semiadditive ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory closed under m-finite p-space (co)limits. Given $X \in \mathcal{C}'$ and $0 \leq n \leq m$ we have

- **(1)** $\mathsf{ht}_{\mathcal{C}',p}(X) \leq n$ if and only if $\mathsf{ht}_{\mathcal{C},p}(X) \leq n$
- **(2)** $\mathsf{ht}_{\mathcal{C},p}(X) > n$ implies $\mathsf{ht}_{\mathcal{C}',p}(X) > n$

Proof.
(1) is immediate from the preservation and reflection of height upper bounds along semiadditive functors.

(2) If $\mathsf{ht}_{\mathcal{C},p}(X) > n$, then for all $Z \in \mathcal{C}_{\leq n}$, $\mathsf{Map}_\mathcal{C}(Z, X) \simeq \mathsf{pt}$. But by **(1)** we have $\mathcal{C}'_{\leq n} = \mathcal{C}_{\leq n} \cap \mathcal{C}'$, and since the inclusion is full, $\mathsf{Map}_{\mathcal{C}'}(A, B) \simeq \mathsf{Map}_\mathcal{C}(A, B)$ for all $A, B \in \mathcal{C}'$. Thus, for all $Z \in \mathcal{C}'_{\leq n}$,

$$\mathsf{Map}_{\mathcal{C}'}(Z, X) \simeq \mathsf{Map}_\mathcal{C}(Z, X) \simeq \mathsf{pt}$$

so that $\mathsf{ht}_{\mathcal{C}',p}(X) > n$.
□

In the case of p-typically m-semiadditively monoidal ∞-categories, the functor perspective on height implies that we can bound the height of the ∞-category via the height of its monoidal unit.

Corollary: ∞-Category Height via Monoidal Unit Height

If $\mathcal{C}$ is a p-typical m-semiadditively monoidal ∞-category and $0 \leq n \leq m$, then $\mathsf{Ht}_p(\mathcal{C}) \leq n$ if and only if $\mathsf{ht}_p(\mathbb{1}) \leq n$.

This follows from the functor perspective on height applied to the p-typically m-semiadditive functors $X \otimes (-) : \mathcal{C} \to \mathcal{C}$ for $X \in \mathcal{C}$.

Comparing Heights

One of the important aspects of semi-additive height is its relation to the classical notion of *stable height*.

> **Stable Height**
>
> For a stable ∞-category $\mathcal{C}$, and $X \in \mathcal{C}$, we define its **p-typical stable height** as follows:
>
> - **(1)** $\mathsf{ht}_{\mathsf{st},p}(X) \leq n$ if $F(n+1) \otimes X \simeq 0$ for some finite p-spectrum $F(n+1)$ of type $(n+1)$.
> - **(2)** $\mathsf{ht}_{\mathsf{st},p}(X) > n$, if $\mathsf{Map}_{\mathcal{C}}(Z, X) \simeq \mathsf{pt}$ for each Z of p-typical stable height $\leq n$

In the case of p-local spectra,

$$\mathsf{Sp}_{(p),\leq^{\mathsf{st}} n} = L_n^f \mathsf{Sp}, \quad \mathsf{Sp}_{(p),>^{\mathsf{st}} n-1} = \mathsf{Sp}_{F(n)}, \quad \mathsf{Sp}_{(p),n^{\mathsf{st}}} = \mathsf{Sp}_{T(n)}$$

A first relation between semiadditive and stable height comes from the following:

> **Lemma: Relation between Semiadditive and Stable Height**
>
> If $\mathcal{C}$ is a stable presentable ∞-semiadditive p-local ∞-category, then for all $0 \leq n, k \leq \infty$, $(\mathcal{C}_{n^{\mathsf{st}}})_k \simeq (\mathcal{C}_k)_{n^{\mathsf{st}}}$.

Idea: We can describe the subcategories of objects at a certain height in terms of tensoring in Pr^L. Further, it turns out that $(\mathcal{C}_{n^{\mathsf{st}}})_n \simeq \mathcal{C}_{n^{\mathsf{st}}}$ and $(\mathcal{C}_{n^{\mathsf{st}}})_k \simeq 0$ for $k \neq n$.

References

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