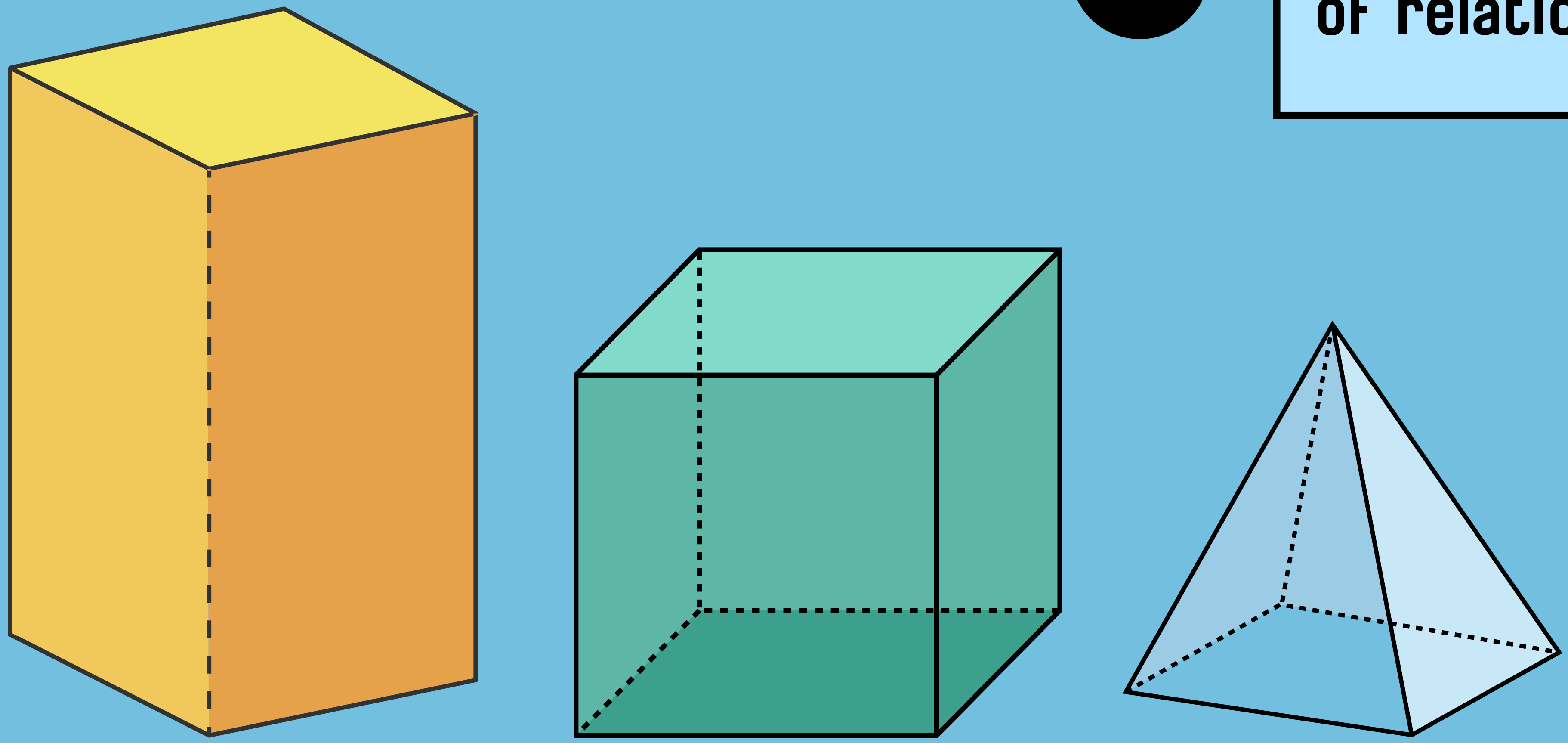
CATEGORY THEORY, YONEDA THEORY, AND FACTORIZATIONS

Abstract Mathematics via Shapes

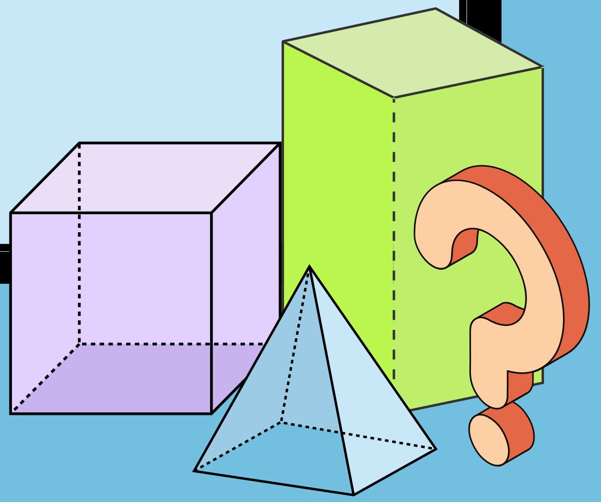
By: Ea E (they/she)
(joint work with Kevin Carlson)

CENTRAL THEMES

1. Category Theory as a relational and compositional perspective on mathematics
2. Higher Category Theories are built from shapes of relations
3. Internalizing Category Theory requires decompositions of relations



WHY CARE ABOUT CATEGORIES?



1. Formal framework for studying patterns and structures in mathematics

2. A language for translating between different mathematical structures

3. Powerful duality statements
   - Tannaka duality (algebraic reconstructions),
   - Isbell duality (algebra-geometry),
   - Gabriel-Ulmer duality

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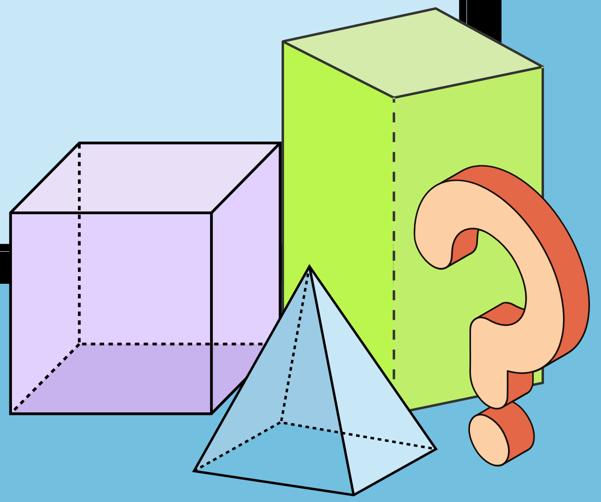


1. Formal framework for studying patterns and structures in mathematics

2. A language for translating between different mathematical structures

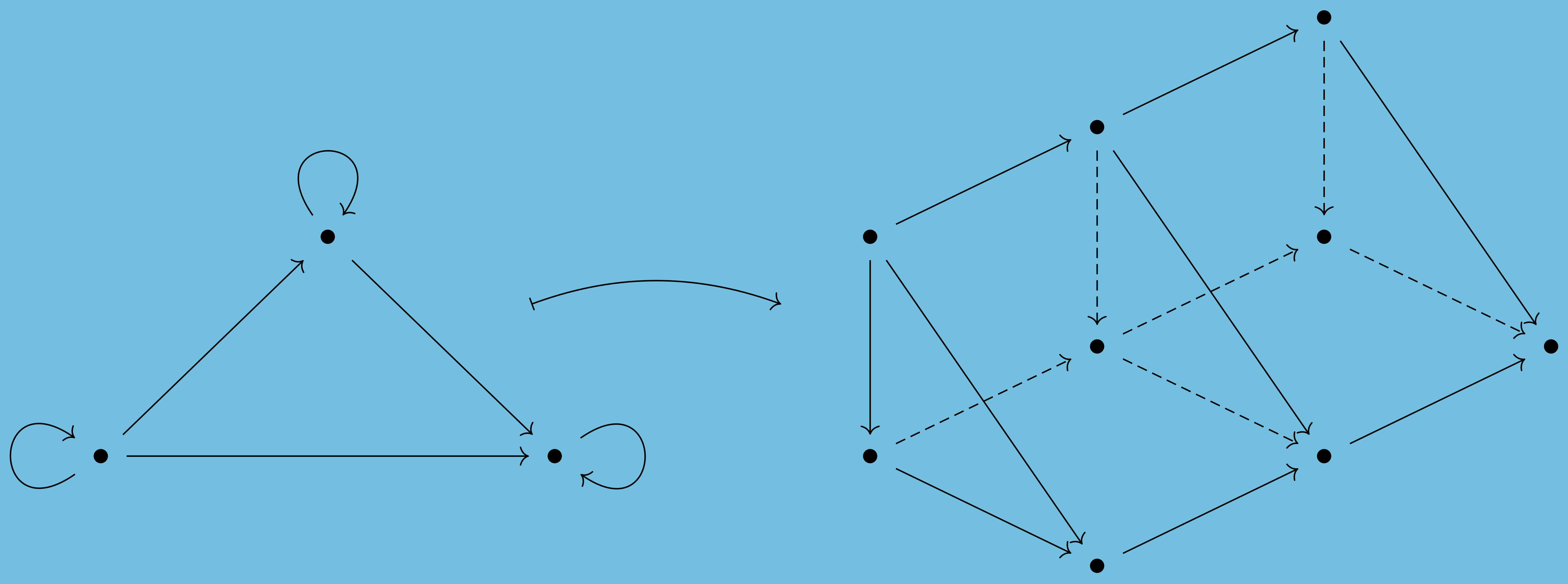
Key to representation theory

3. Powerful duality statements
   - Tannaka duality (algebraic reconstructions),
   - Isbell duality (algebra-geometry),
   - Gabriel-Ulmer duality

E.g. Commutative C*-algebras vs compact Hausdorff Spaces

THE WHAT AND WHY OF CATEGORIES

Key Philosophy: Structure is detected through relations



THE WHAT AND WHY OF CATEGORIES

The Data of a Category

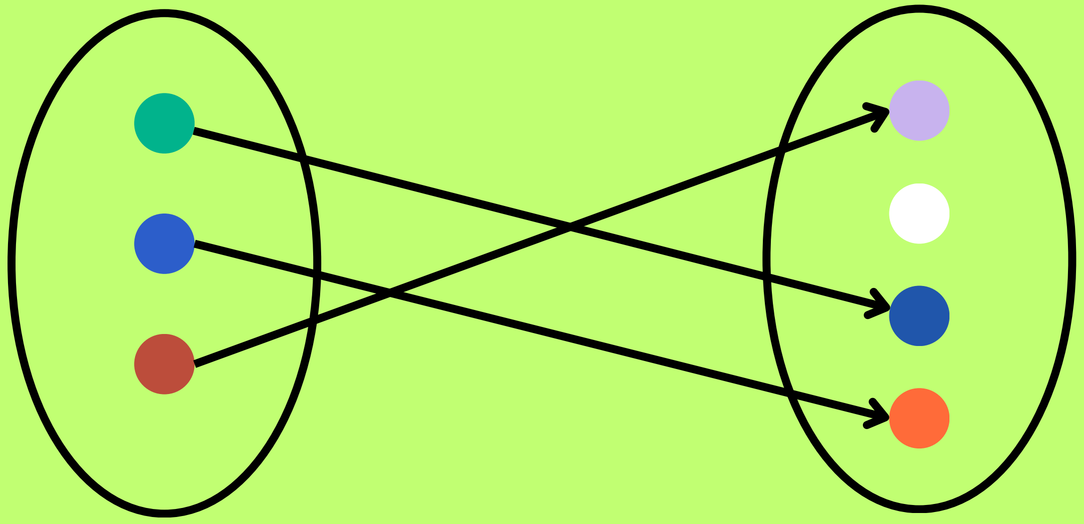
- Objects, a,b,c,d,...
- Maps/arrows/relations between objects,
- An operation for composing relations
- A distinguished identity operation for each object

$$
\begin{tikzcd}
 & b \arrow[loop above, "\mathrm{id}_b"] \arrow[dr, "g"] & \\
a \arrow[loop left, "\mathrm{id}_a"] \arrow[ur, "f"] \arrow[rr, "f;g"'] & & c \arrow[loop right, "\mathrm{id}_c"]
\end{tikzcd}
$$

THE WHAT AND WHY OF CATEGORIES

Examples of Categories

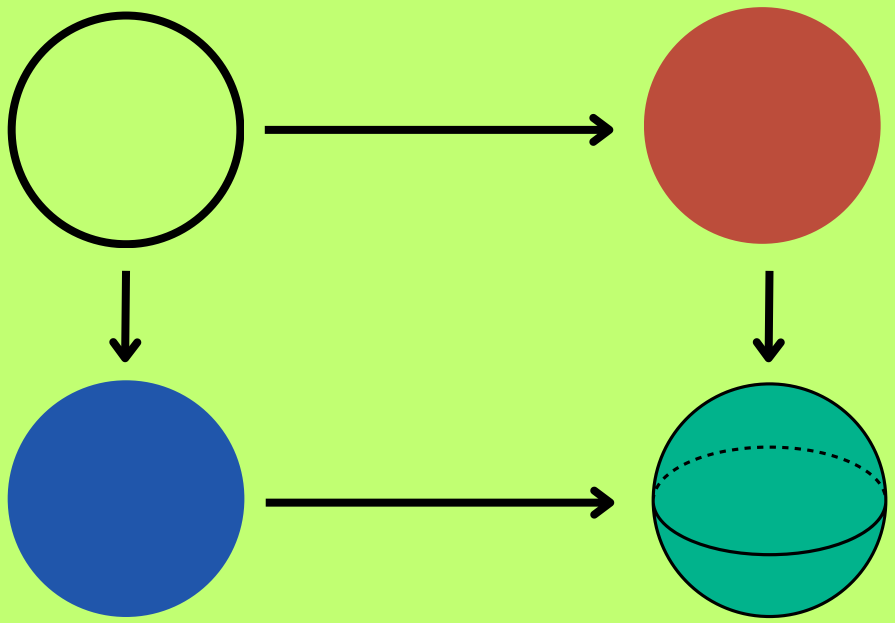
Set



Z

$$\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

Top



Grp

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] & \mathbb{Z} \arrow[l, dashed, bend left, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[r] & 0 \end{tikzcd}$$

FUNCTORS: THE RELATIONS BETWEEN CATEGORIES

Key Idea: Functors allow us to transfer information between mathematical universes

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Key Idea: Functors allow us to transfer information between mathematical universes

Functors map the data of one category to another while respecting compositions

Examples of Functors

Free-Forgetful Functors

$$\begin{tikzcd} \mathsf{Ab} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

(Co)Homology

$$\mathsf{Top} \xrightarrow{H_*} \mathsf{Gr}(\mathsf{Ab})$$

$$\mathsf{Top}^{op} \xrightarrow{H^*} \mathsf{Gr}(\mathsf{Ab})$$

Products

$$\mathsf{C} \xrightarrow{A\times -} \mathsf{C}$$

$$B \longmapsto A \times B$$

Maps

$$\mathsf{C}^{op} \times \mathsf{C} \xrightarrow{\mathsf{Map}_{\mathsf{C}}(-,-)} \mathsf{Set}$$

$$(A, B) \longmapsto \mathsf{Map}_{\mathsf{C}}(A, B)$$

NATURAL TRANSFORMATIONS: THE RELATIONS BETWEEN FUNCTORS

Key Idea: Natural transformations relate transfers of information between categories

Two perspectives on Natural Transformations

- Natural transformations are thickened functors

$$
\begin{tikzcd}
\mathsf{C} \arrow[d, "s"'] \arrow[drr, "F"] & & \\
\mathsf{C} \times \downarrow \arrow[rr, "\alpha"] & & \mathsf{D} \\
\mathsf{C} \arrow[u, "t"] \arrow[urr, "G"'] & &
\end{tikzcd}
$$

- Natural transformations intertwine between functors

$$
\begin{tikzcd}
F(a) \arrow[r, "F(f)"] \arrow[d, "\alpha_a"'] \arrow[dr, phantom, "\circlearrowright"] & F(b) \arrow[d, "\alpha_b"] \\
G(a) \arrow[r, "G(f)"'] & G(b)
\end{tikzcd}
$$

NATURAL TRANSFORMATIONS: THE RELATIONS BETWEEN FUNCTORS

Key Idea: Natural transformations relate transfers of information between categories

Key Consequence: We get an internal category of maps

Two perspectives on Natural Transformations

- Natural transformations are thickened functors

$$\begin{tikzcd} \mathsf{C} \arrow[d, "s"'] \arrow[drr, "F"] & & \\ \mathsf{C}\times\downarrow \arrow[rr, "\alpha" description] & & \mathsf{D} \\ \mathsf{C} \arrow[u, "t"] \arrow[urr, "G"'] & & \end{tikzcd}$$

- Natural transformations intertwine between functors

$$\begin{tikzcd} F(a) \arrow[rr, "F(f)"] \arrow[d, "\alpha_a"'] & & F(b) \arrow[d, "\alpha_b"] \\ G(a) \arrow[rr, "G(f)"'] \arrow[urr, phantom, "\circlearrowright"] & & G(b) \end{tikzcd}$$

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D})$$

Ob : $\mathsf{C} \xrightarrow{F} \mathsf{D}$

Map :

$$\begin{tikzcd} \mathsf{C} \arrow[d, "s"'] \arrow[drr, "F"] & & \\ \mathsf{C}\times\downarrow \arrow[rr, "\alpha" description] & & \mathsf{D} \\ \mathsf{C} \arrow[u, "t"] \arrow[urr, "G"'] & & \end{tikzcd}$$

THE YONEDA EMBEDDING!

For a category C, an object c is fully determined by the sets Map(d,c) where d ranges over the objects of C.

$$\mathsf{C} \xhookrightarrow{\mathsf{Yoneda}} \mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$$

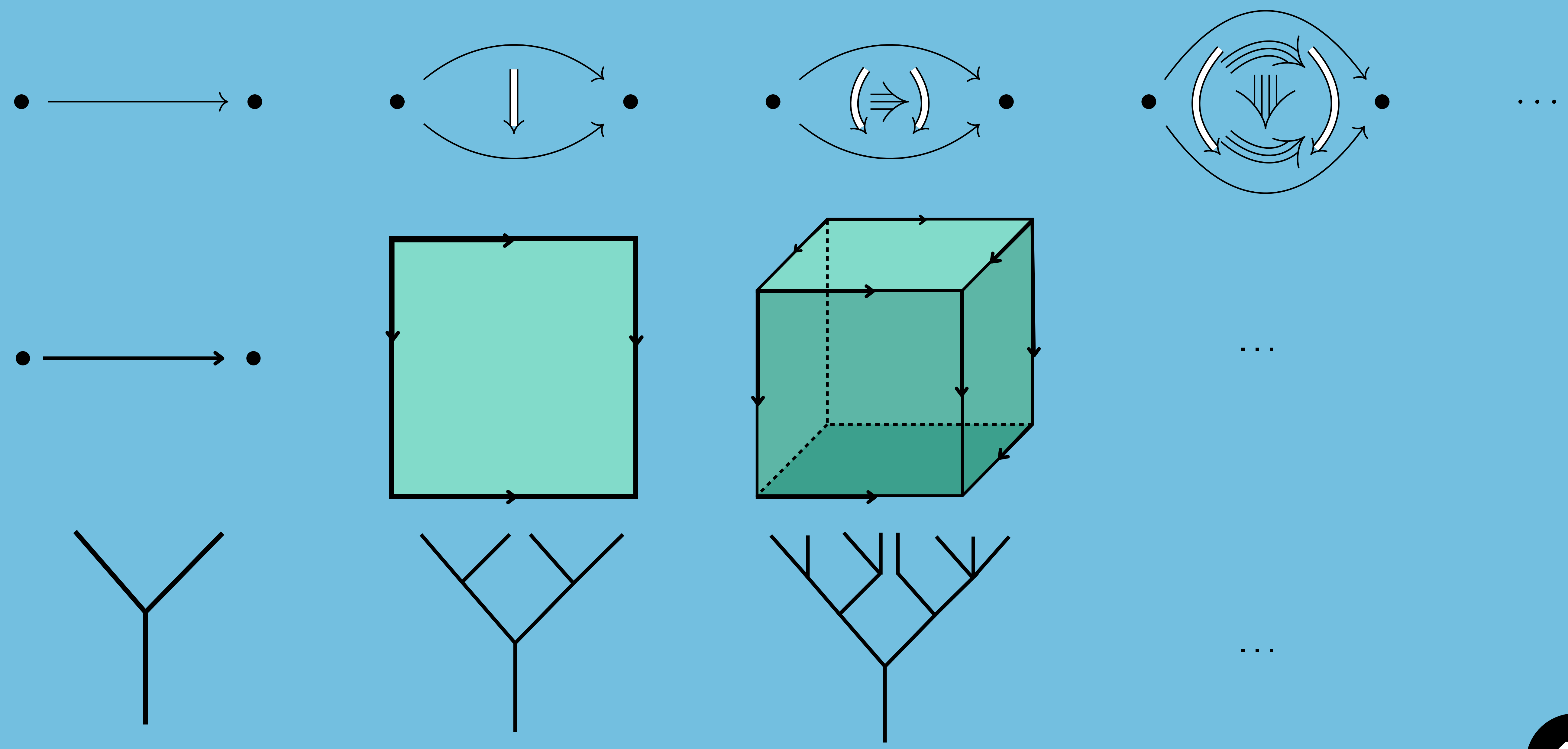
$$A \longmapsto \mathsf{Map}_{\mathsf{C}}(-, A)$$

Dually, the object c is fully determined by the sets Map(c,d) where d again ranges over the objects of C.

HIGHER CATEGORIES AS SHAPE INDEXED SETS/SPACES

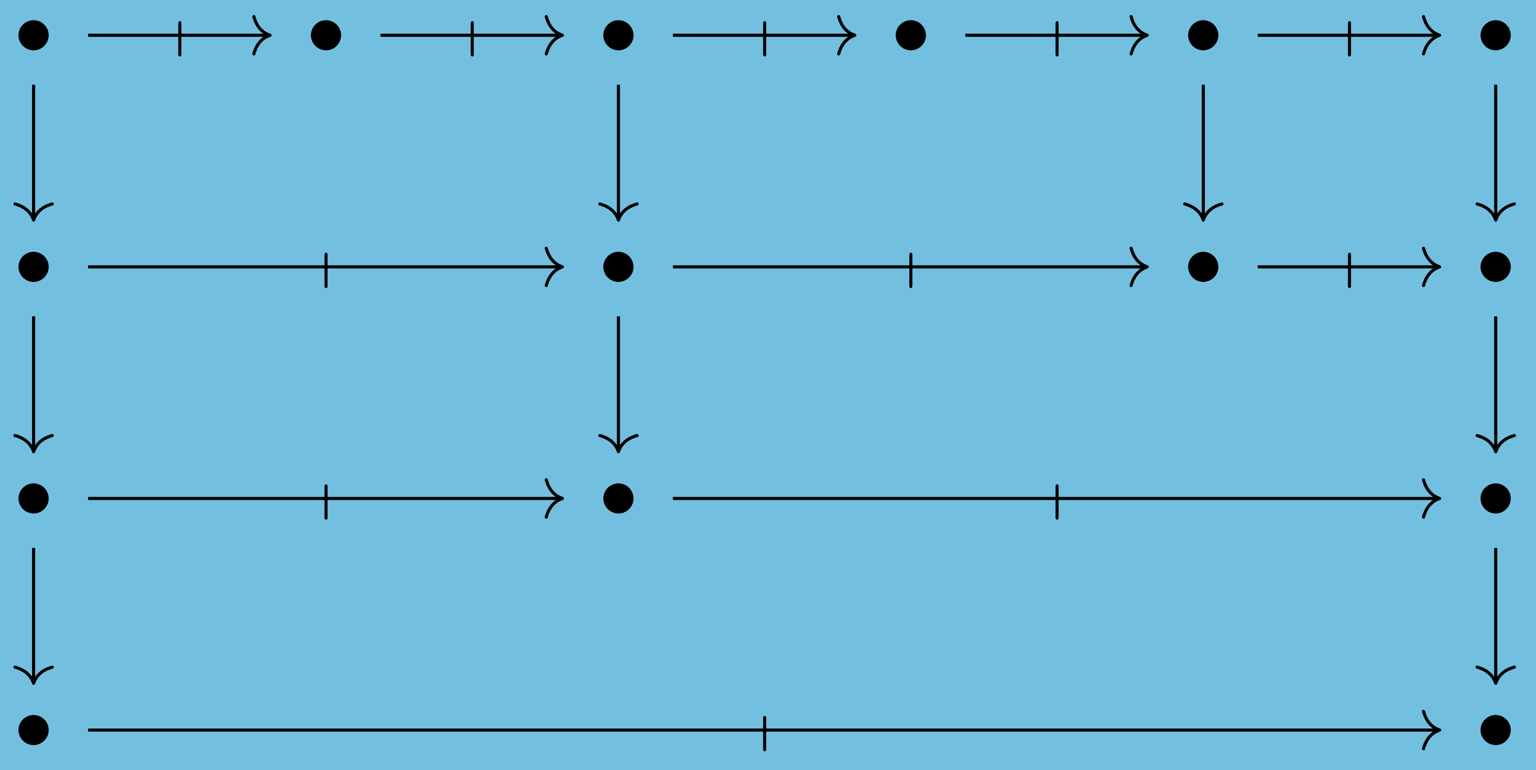
SHAPES OF HIGHER RELATIONS

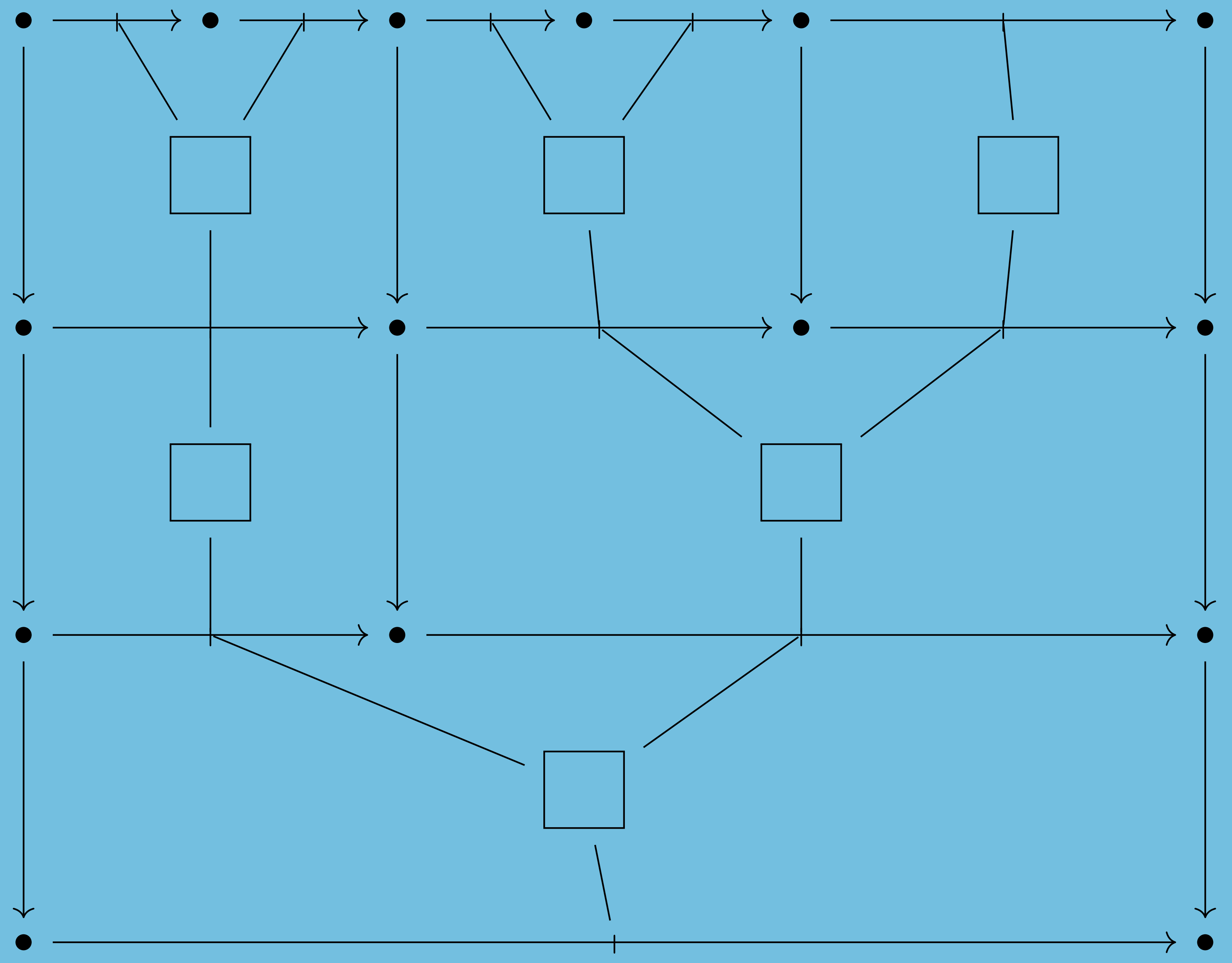
- Globular:
- Cubical:
- Trees:



SHAPES OF HIGHER RELATIONS

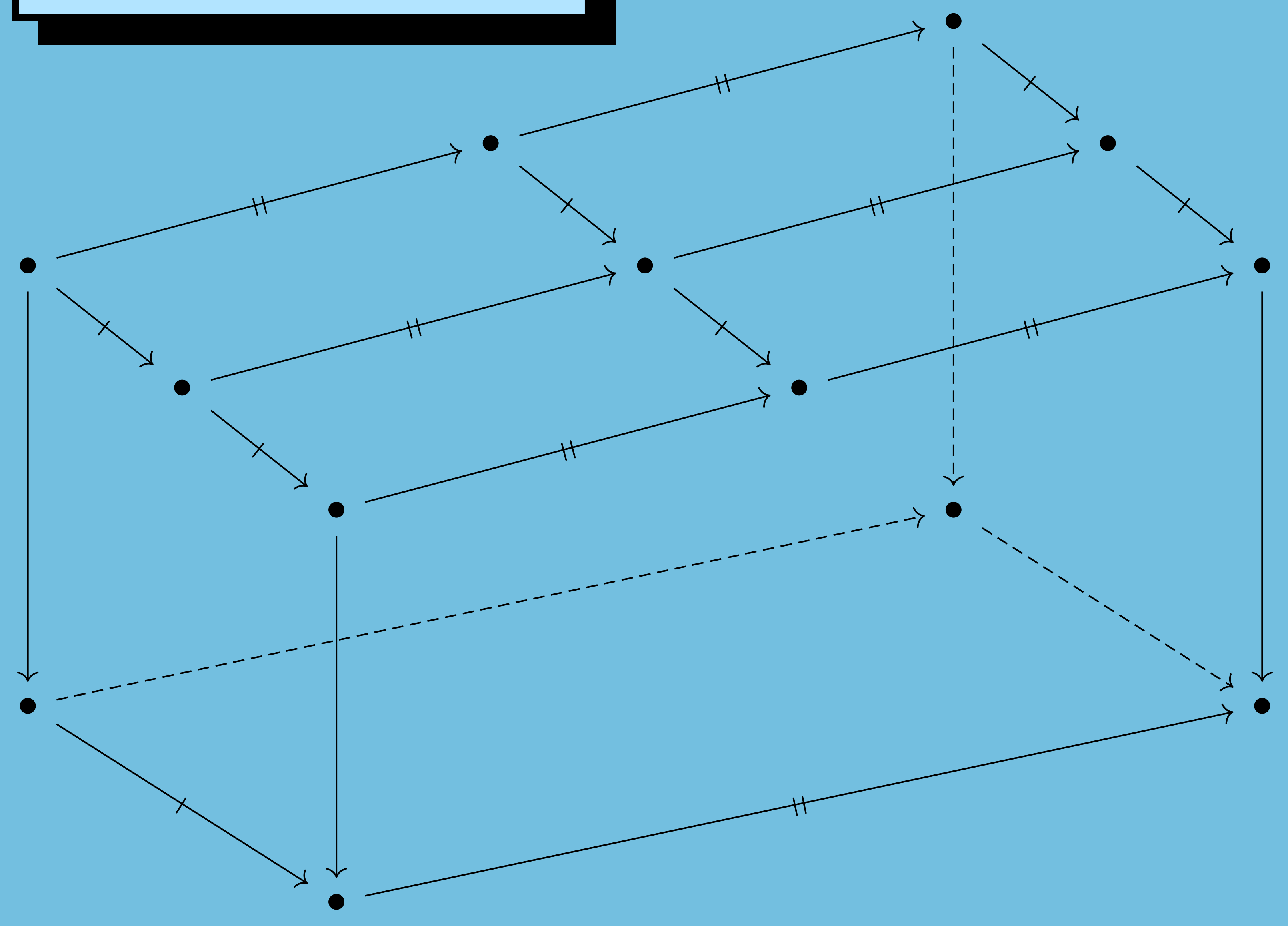
Globular Trees

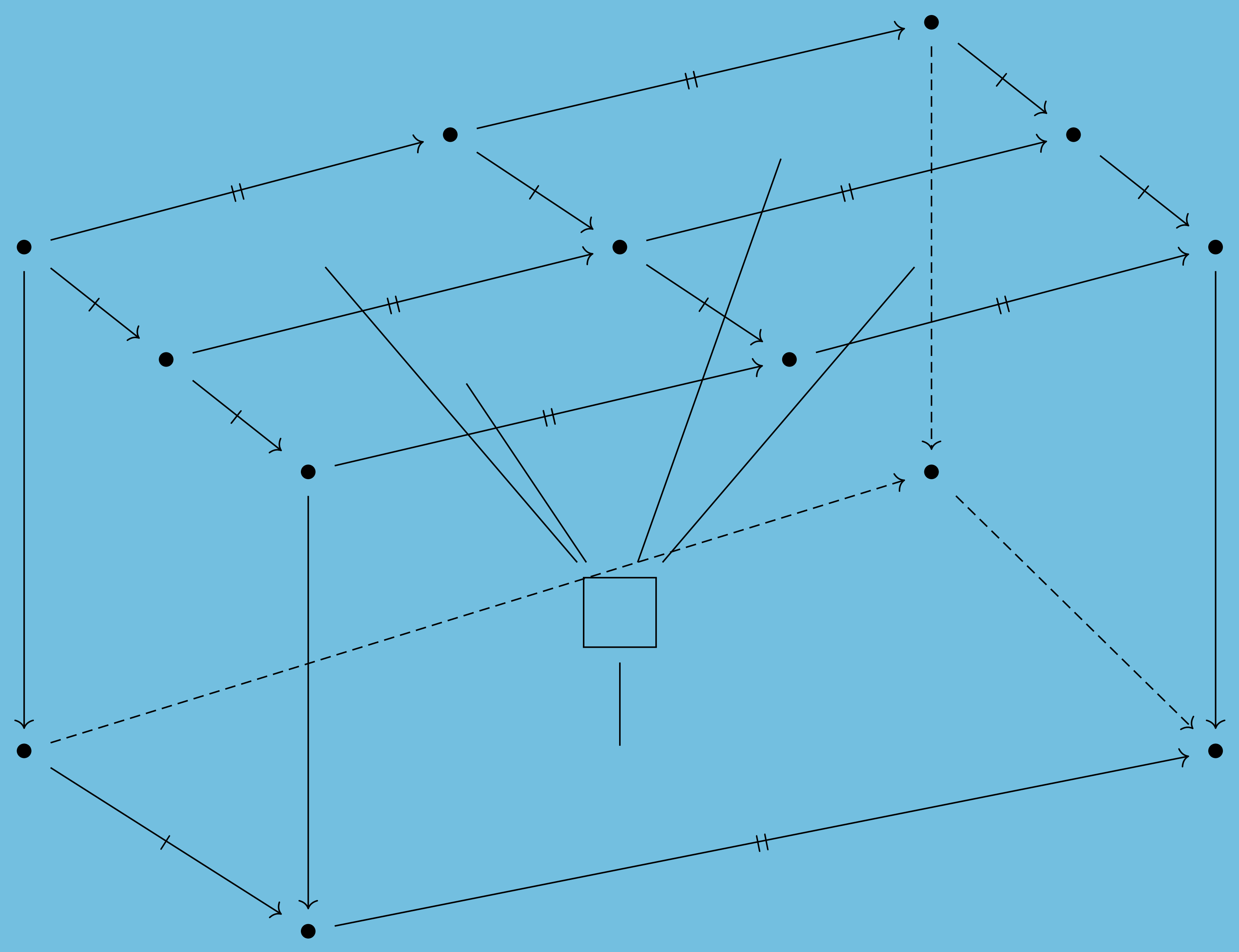




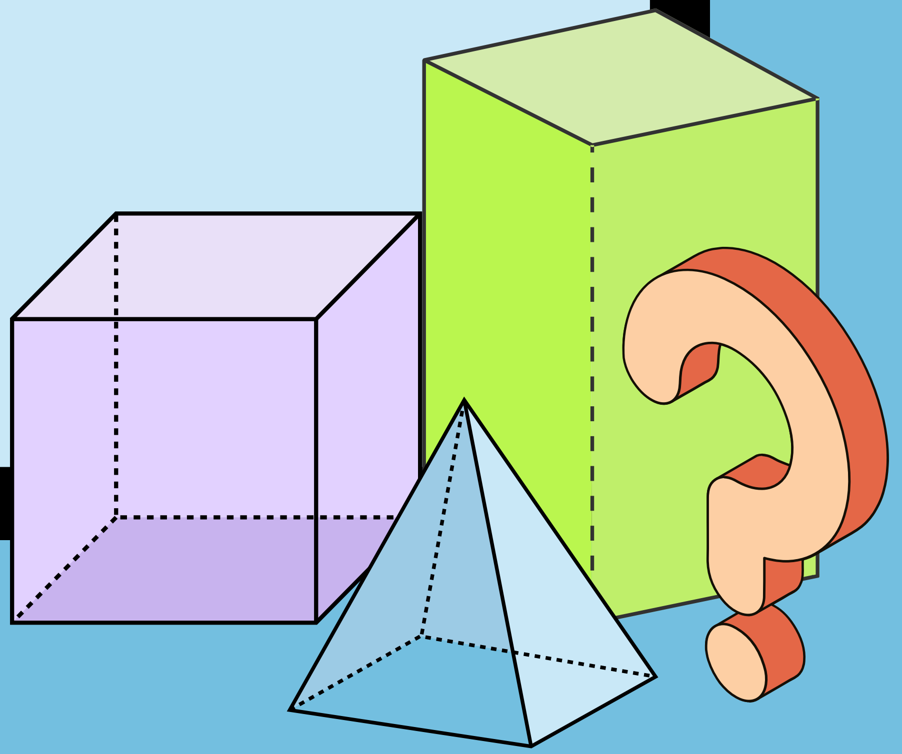
SHAPES OF HIGHER RELATIONS

Cubical Trees





DO THESE SHAPED CATEGORY THEORIES ADMIT INTERNAL YONEDA EMBEDDINGS?



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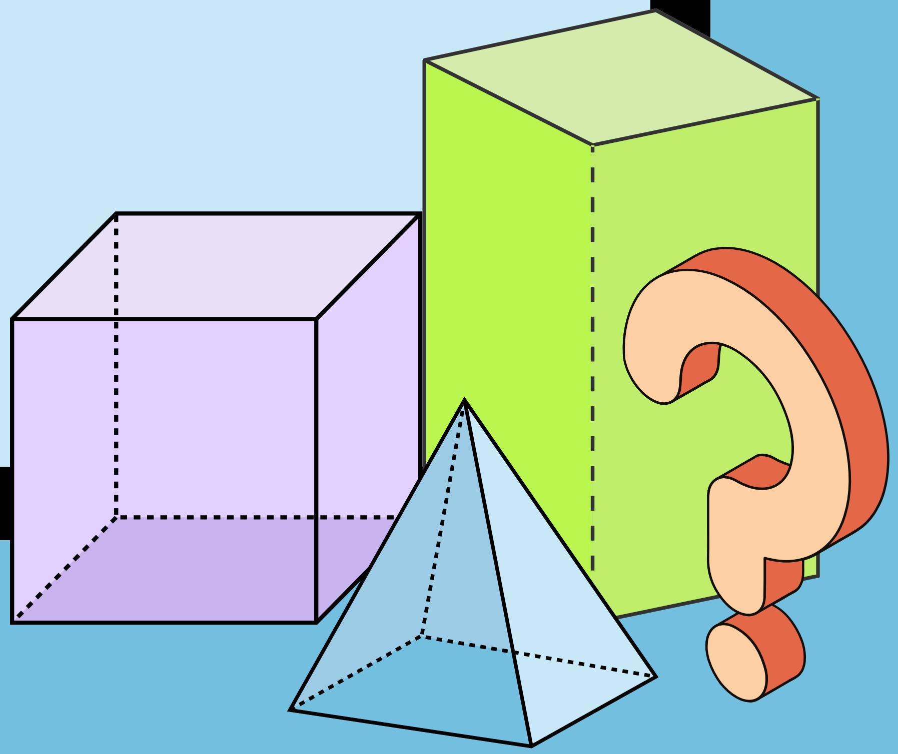


DISTINCTION BETWEEN SHAPES

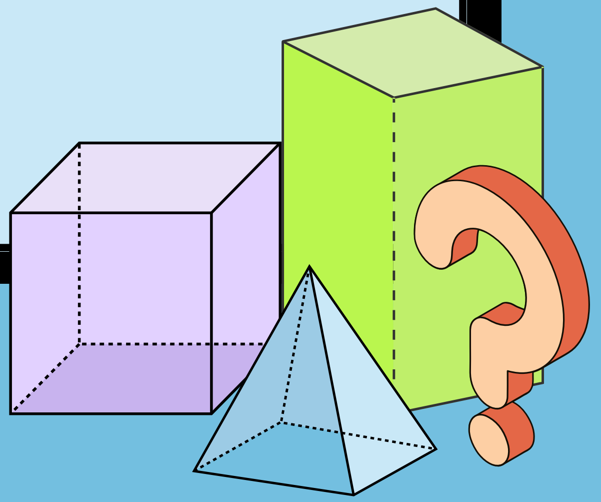
Shapes Admitting Yoneda Embeddings

- Sets
- Globular higher categories
- Cubical higher categories

Shapes Not Admitting Yoneda Embeddings

- Trees
- Globular trees
- Cubical trees

HOW DO GLOBULAR TREES FAIL?

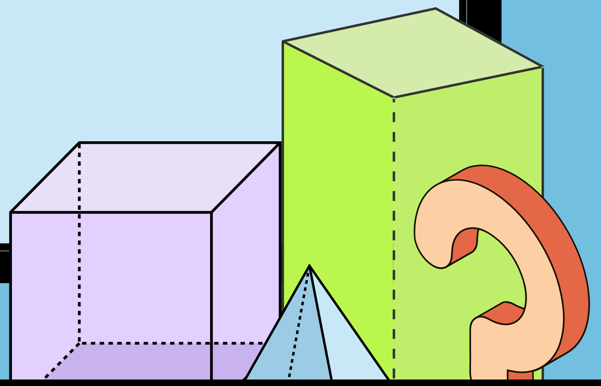


The Data of a 1-Globular Multicategory

- Objects, a,b,c,d,...
- Tight arrows $\downarrow$ that can be composed
- Loose arrows $\begin{tikzcd} {} \arrow[r, "\shortmid" marking] & {} \end{tikzcd}$
- Multicells that can be pasted

$$\begin{tikzcd} \bullet \arrow[r, "\shortmid" marking] \arrow[d] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\ \bullet \arrow[rrr, "\shortmid" marking] & & & \bullet \end{tikzcd}$$

HOW DO GLOBULAR TREES FAIL?



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- Objects, a,b,c,d,...
- Tight arrows $\downarrow$ that can be composed
- Loose arrows $\mathrel{\mapstochar\rightarrow}$
- Multicells that can be pasted

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid" marking] & & & \bullet
\end{tikzcd}
$$

Multicells of An Internal Mapping Object

For 1-globular multicategories D and E, multicells in Map(D,E) are assignments of multicells for specified shapes:

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[drr, phantom, "\alpha"] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\
\bullet \arrow[rr, "\shortmid" marking] & & \bullet
\end{tikzcd}
\xmapsto{\Gamma}
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[drr, phantom, "\Gamma(\alpha)"] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\
\bullet \arrow[rr, "\shortmid" marking] & & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[drrr, phantom, "\beta"] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid" marking] & & & \bullet
\end{tikzcd}
\xmapsto{\Phi}
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[drrrr, phantom, "\Phi(\beta)"] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid" marking] & & & & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\delta"] & \bullet \arrow[d] \\
\bullet \arrow[r, "\shortmid" marking] & \bullet
\end{tikzcd}
\xmapsto{\Psi}
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Psi(\delta)"] & \bullet \arrow[d] \\
\bullet \arrow[r, "\shortmid" marking] & \bullet
\end{tikzcd}
$$

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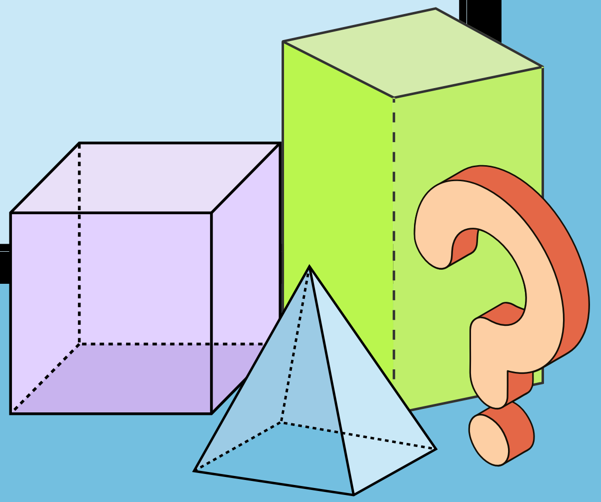


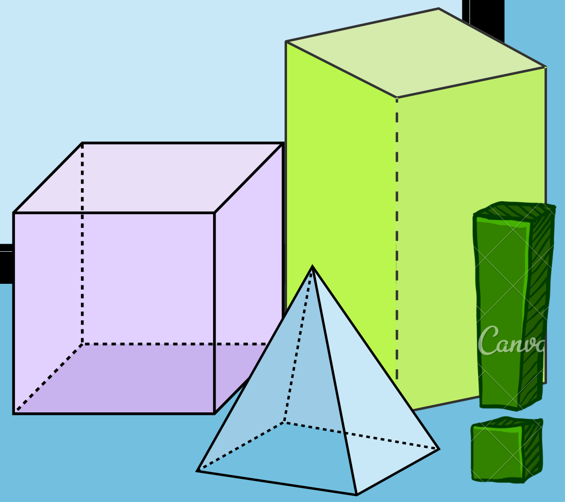
Composing Multicells!

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drrr, phantom, "\Phi"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\Psi"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid"{marking}] \arrow[d] \arrow[drrrr, phantom, "\Gamma"] & & & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drrrr, phantom, "?"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
$$

Note: A priori we can't define an action on 4-to-1 multicells

WHAT'S MISSING FOR INTERNAL MAPPING OBJECTS?

DECOMPOSITIONS PROVIDE INTERNAL MAPPING OBJECTS!

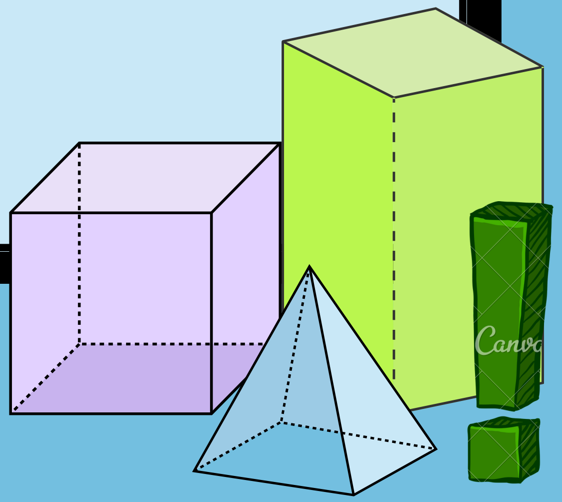


Consider mapping multicells

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drrr, phantom, "\Phi"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\Psi"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid"{marking}] \arrow[d] \arrow[drrrr, phantom, "\Gamma"] & & & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[dd] \arrow[ddrrrr, phantom, "?"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[dd] \\
& & & & \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[dd] \arrow[ddrrrr, phantom, "\alpha"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[dd] \\
& & & & \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drrr, phantom, "\alpha_1"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid"{marking}] \arrow[d] \arrow[drrrr, phantom, "\alpha_3"] & & & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drrr, phantom, "\Phi(\alpha_1)"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\Psi(\alpha_2)"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid"{marking}] \arrow[d] \arrow[drrrr, phantom, "\Gamma(\alpha_3)"] & & & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
$$

DECOMPOSITIONS PROVIDE INTERNAL MAPPING OBJECTS!



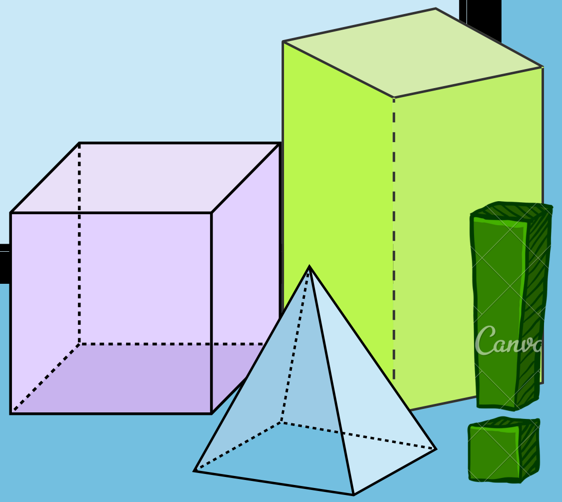
Thm: Mapping Objects from Decompositions

For any collection of indexing shapes S, if C is an S shaped higher category, then internal mapping objects out of C,

$$\underline{\mathrm{Map}}_{\mathrm{Cat}_S}(\mathrm{C}, \mathrm{D}) \in \mathrm{Cat}_S$$

exist if and only if C admits essentially unique decompositions of its S-shaped relations

DECOMPOSITIONS PROVIDE INTERNAL MAPPING OBJECTS!



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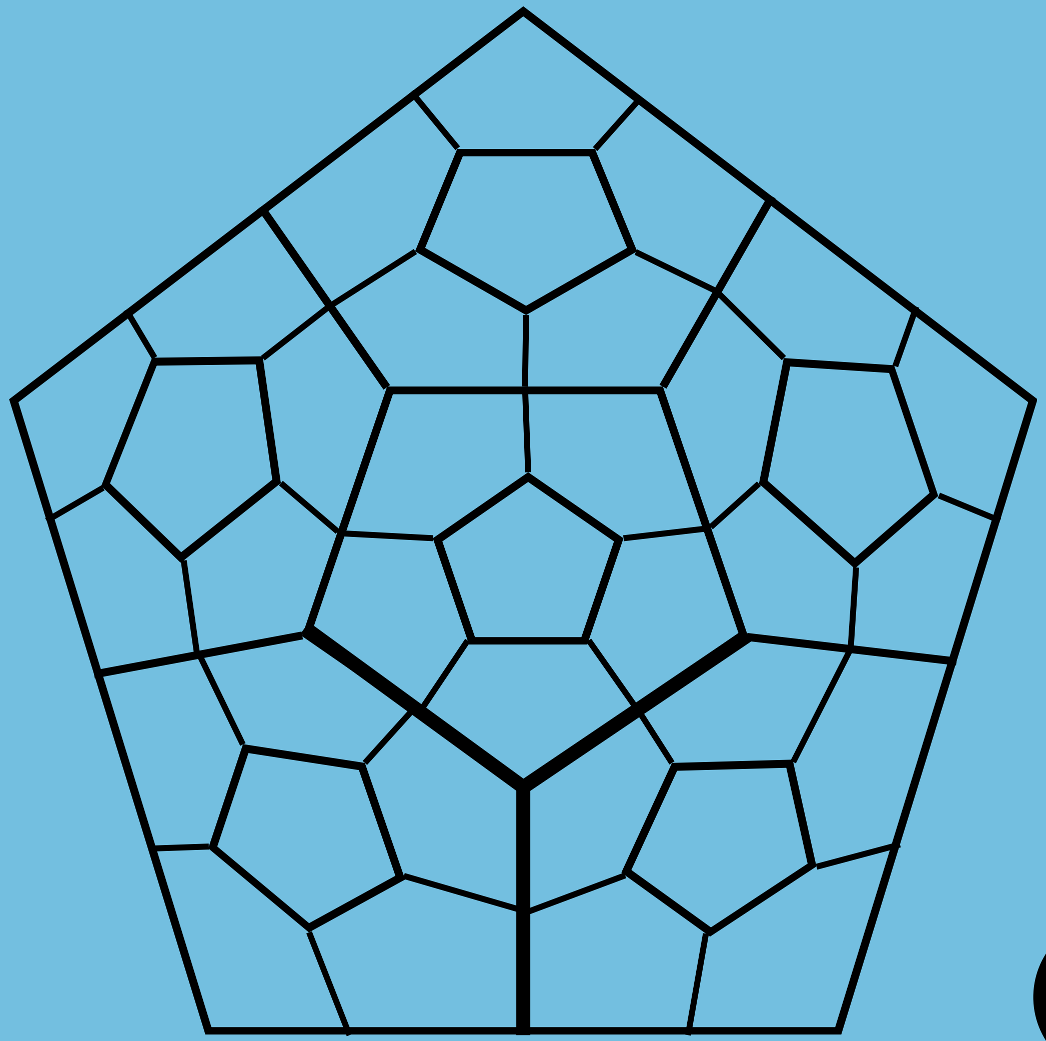
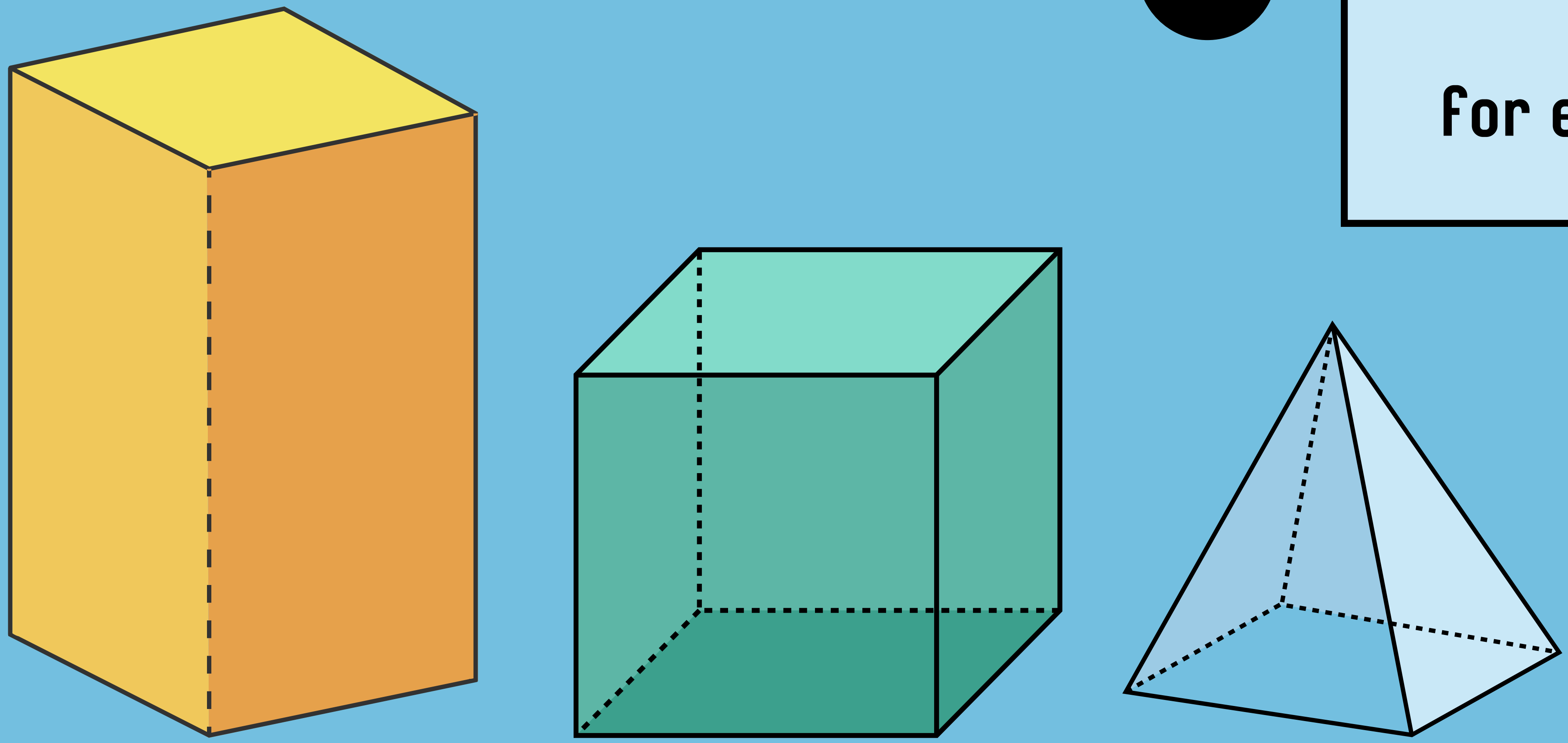
Tilings=Compositional Relations[2]



LET'S RECAP

1. Category theory provides a relational approach to mathematics, justified by the Yoneda lemma

2. Higher categories are parameterized by shapes, and Yoneda embeddings in higher contexts require asking for essentially unique decompositions of relations



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