EXPONENTIABLE VIRTUAL DOUBLE CATEGORIES

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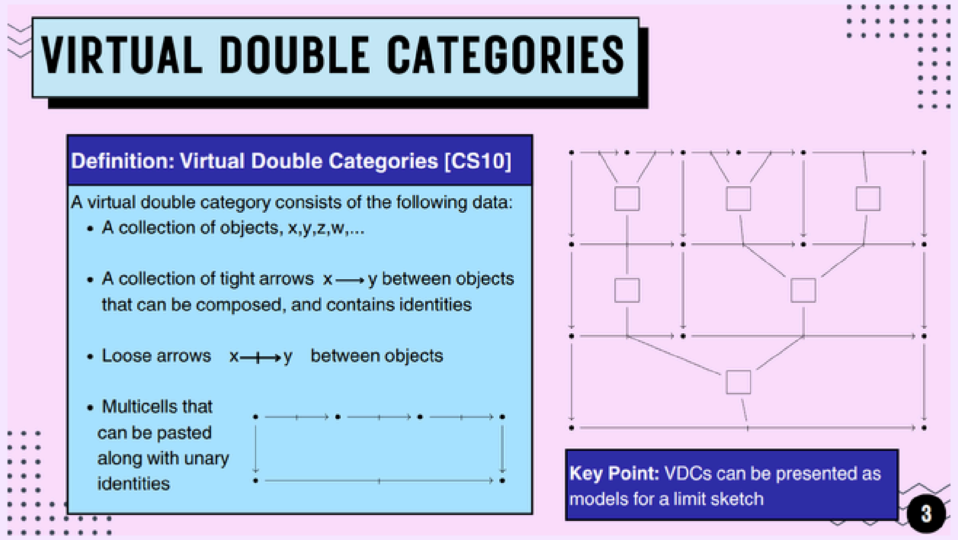
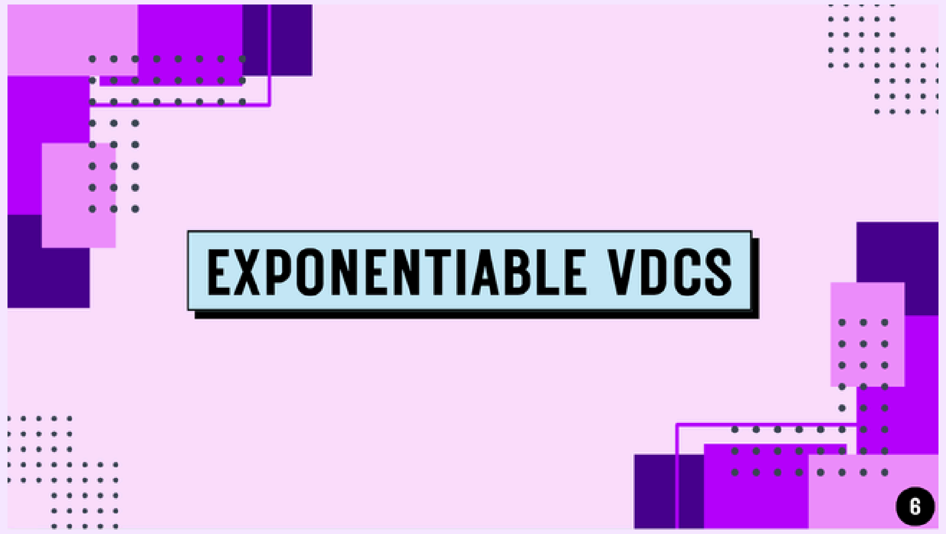
ROADMAP

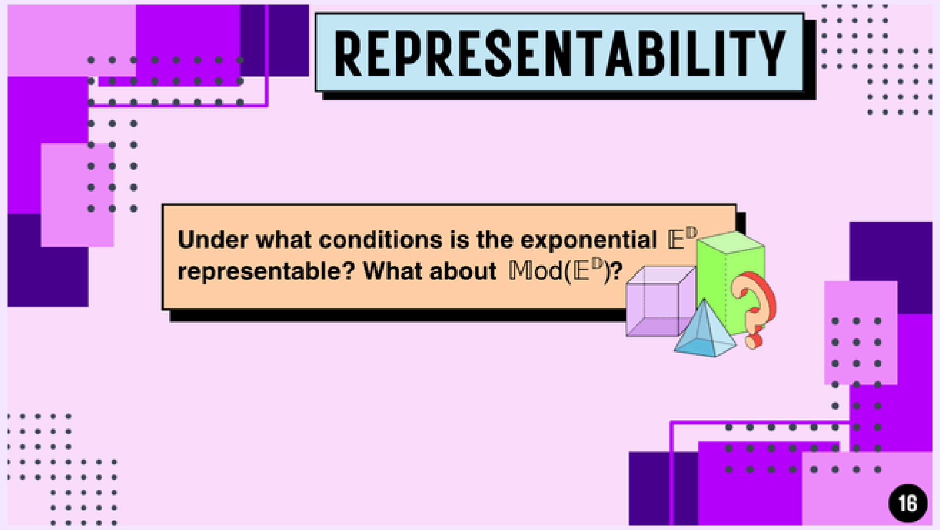
Intro to VDCs:



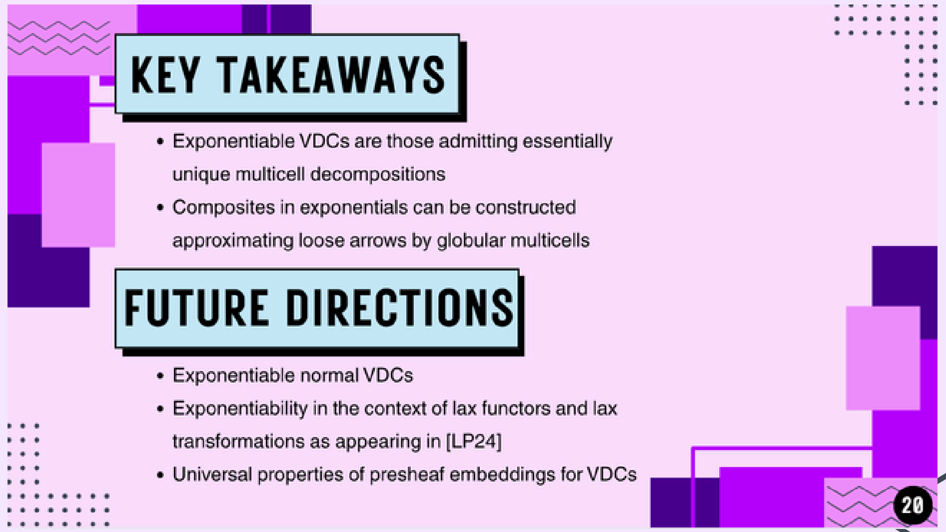
Exponentiability:



Representability:



Summary and Future Directions:



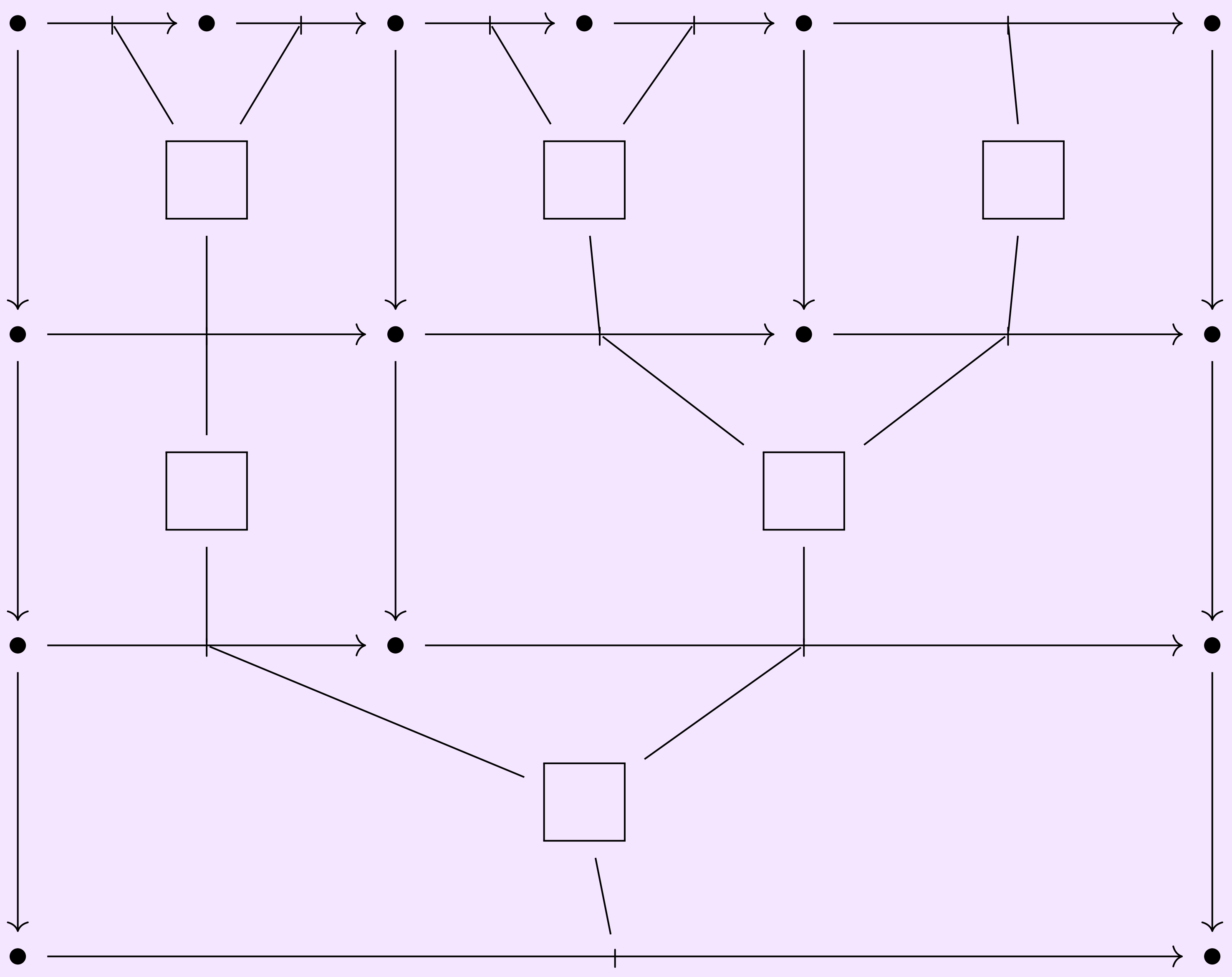
VIRTUAL DOUBLE CATEGORIES

Definition: Virtual Double Categories [CS10]

A virtual double category consists of the following data:

- A collection of objects, x,y,z,w,...
- A collection of tight arrows $x \longrightarrow y$ between objects that can be composed, and contains identities
- Loose arrows $x \mathrel{\mapstochar\rightarrow} y$ between objects
- Multicells that can be pasted along with unary identities

$$\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[d] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "|" marking] & & & \bullet
\end{tikzcd}$$



Key Point: VDCs can be presented as models for a limit sketch

CATEGORIFICATION

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{StrictMonCat} \arrow[r, phantom, "\perp"] \arrow[r, hook, bend right, "B"'] \arrow[d, hook] & \mathbf{2Cat} \arrow[l, bend right, "\mathsf{Coll}_0"'] \arrow[r, hook] \arrow[d, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{Dbl} \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "\mathbb{O}\mathsf{b}_0"] \arrow[d, hook] \\
\mathbf{Multcat} \arrow[r, phantom, "\perp"] \arrow[r, hook, bend right, "B"'] & \mathbf{Vbc} \arrow[l, bend right, "\mathsf{Coll}_0"'] \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{Vdc} \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "\mathbb{O}\mathsf{b}_0"]
\end{tikzcd}
$$

EXAMPLES AND CONSTRUCTIONS

Examples:

- For any category $\mathscr{E}$ we have a VDC $\mathbb{C}\mathrm{ospan}(\mathscr{E})$ whose loose arrows are cospans in $\mathscr{E}$, which is representable if and only if $\mathscr{E}$ has pushouts;
- For any VDC $\mathbb{D}$, we have a VDC $\mathbb{M}\mathrm{od}(\mathbb{D})$ of loose monads in $\mathbb{D}$;
- For any VDC $\mathbb{D}$, we have a VDC $\mathbb{D}\mathbb{M}\mathrm{at}$ of families $\mathbb{D}$, and a VDC $\mathbb{D}\mathbb{P}\mathrm{rof} = \mathbb{M}\mathrm{od}(\mathbb{D}\mathbb{M}\mathrm{at})$ of categories enriched in $\mathbb{D}$;
- For any pair of pseudo-double categories $\mathbb{D}$ and $\mathbb{E}$, we have a VDC $\mathbb{L}\mathrm{ax}(\mathbb{D},\mathbb{E})$ of lax double functors, tight transformations, and modulations.

EXPONENTIABLE VDCS

EXPONENTIABLE VDCS

Definition: Exponentiable VDCs

A virtual double category is said to be exponentiable if its product functor admits a right adjoint:

$$\begin{tikzcd} \mathsf{Vdc} \arrow[rr, bend left, "- \times \mathbb{D}"] \arrow[rr, phantom, "\perp"] & & \mathsf{Vdc} \arrow[ll, bend left, "(-)^{\mathbb{D}}"] \end{tikzcd}$$

EXPONENTIAL DATA

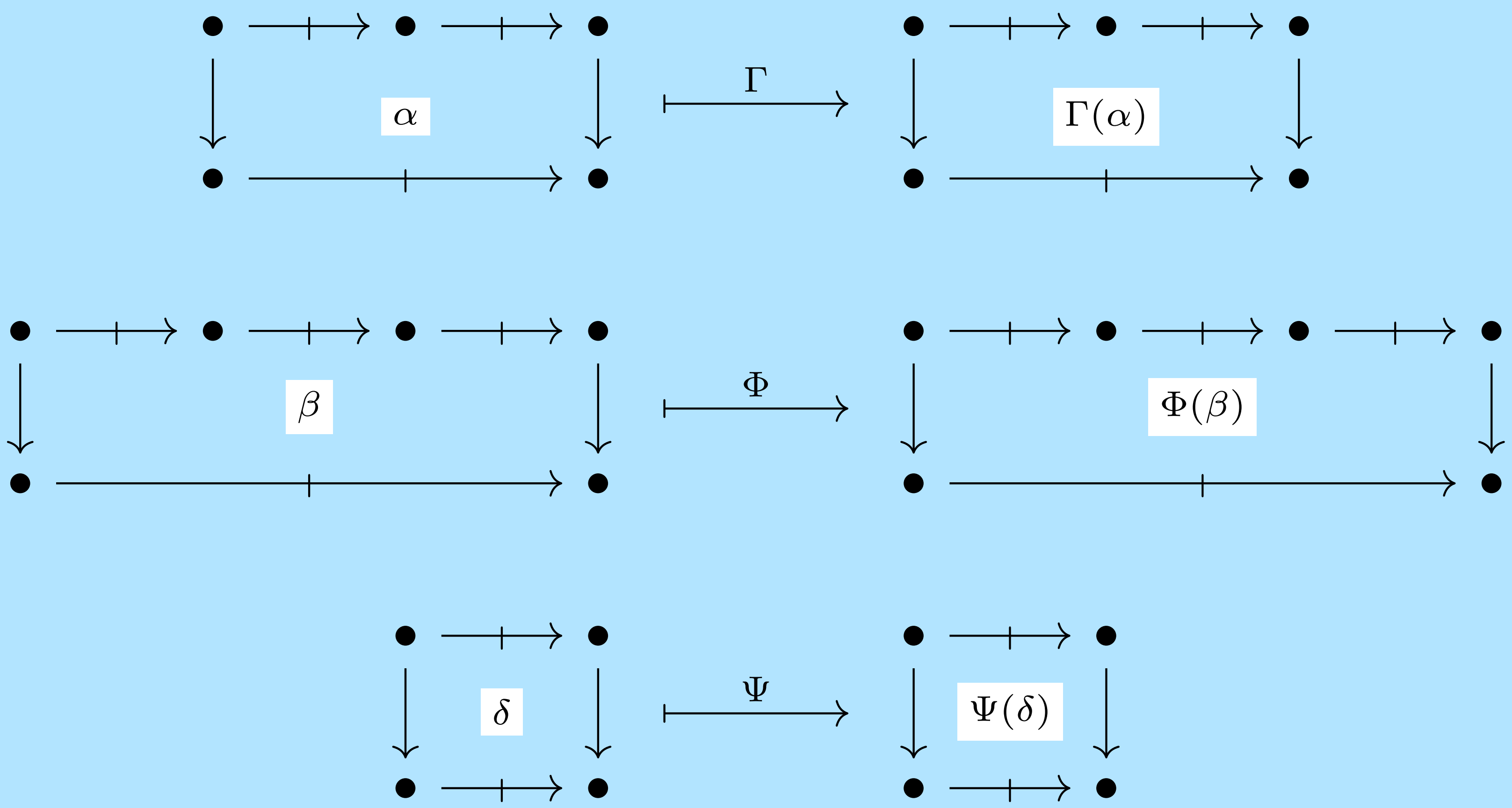
Explication: Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then it consists of the following data:

- Objects are functors on underlying tight categories;
- Tight arrows are natural transformations between such functors;
- Loose arrows are spans of functors:

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[d, "F_0"'] & \mathbb{D}_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\Phi", dashed] & \mathbb{D}_0 \arrow[d, "F_1"] \\
\mathbb{E}_0 & \mathbb{E}_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathbb{E}_0
\end{tikzcd}$$

- Multicells are assignments on multicells of the specified shape, that are functorial under pasting with unary multicells:



HOW DOES EXPONENTIABILITY FAIL?

Composing Multicells!

$$
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[drrr, phantom, "\Phi"] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Psi"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid" marking] \arrow[d] \arrow[drrrr, phantom, "\Gamma"] & & & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid" marking] & & & & \bullet
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[dd] \arrow[ddrrrr, phantom, "?"] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[dd] \\
& & & & \\
\bullet \arrow[rrrr, "\shortmid" marking] & & & & \bullet
\end{tikzcd}
$$

Note: A priori we can't define an action on quaternary multicells

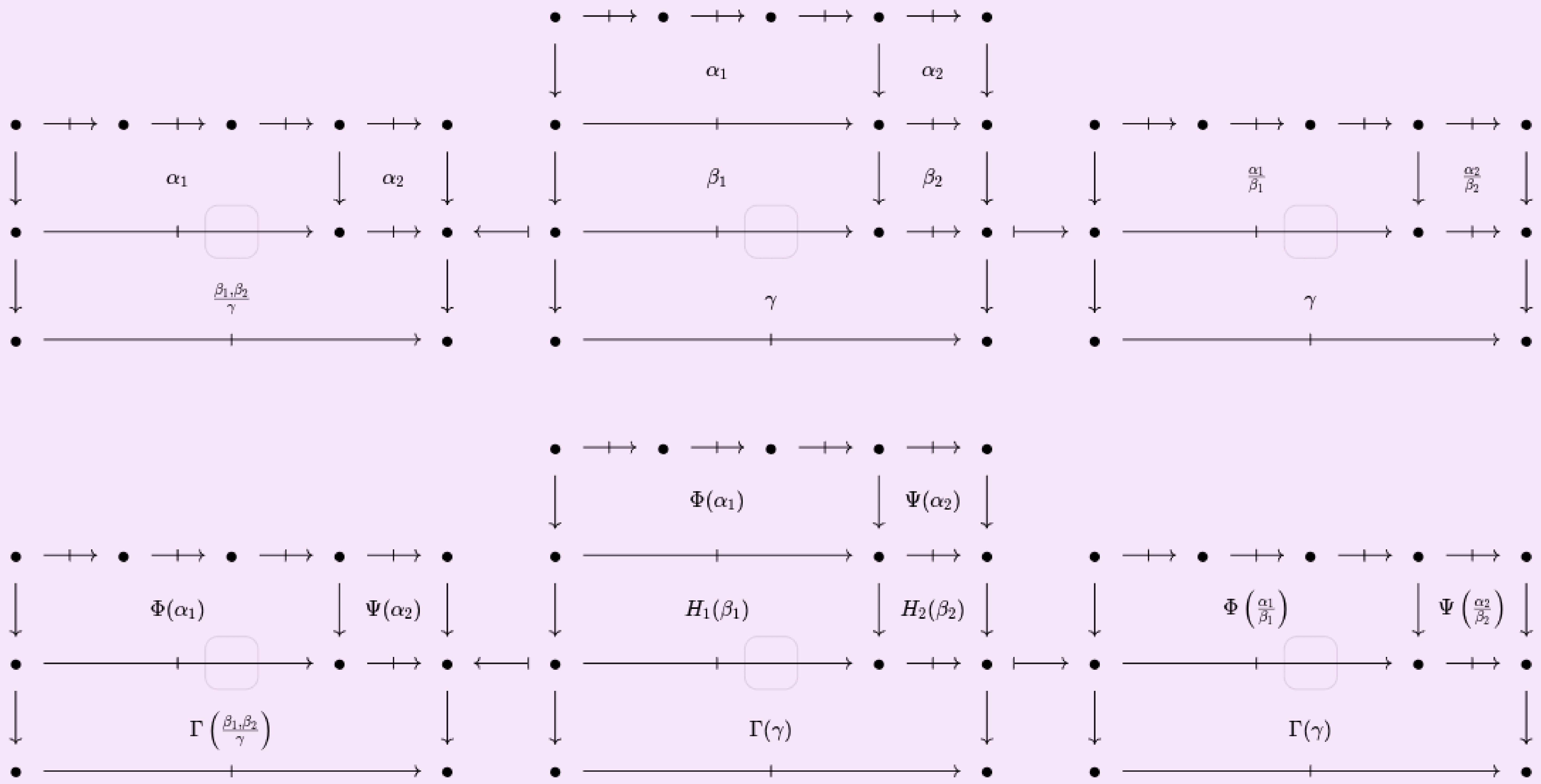
SOLUTION: WE NEED DECOMPOSITIONS OF MULTICELLS!

Consider mapping multicells:

$$
\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[d] \arrow[rrr, phantom, bend left=0, shift right=6ex, "\Phi"] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] \arrow[d] \arrow[r, phantom, shift right=6ex, "\Psi"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "|" marking] \arrow[d] \arrow[rrrr, phantom, shift right=6ex, "\Gamma"] & & & \bullet \arrow[r, "|" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "|" marking] & & & & \bullet
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[dd] \arrow[rrrr, phantom, shift right=12ex, "?"] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[dd] \\
& & & & \\
\bullet \arrow[rrrr, "|" marking] & & & & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[dd] \arrow[rrrr, phantom, shift right=12ex, "\alpha"] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[dd] \\
& & & & \\
\bullet \arrow[rrrr, "|" marking] & & & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[d] \arrow[rrr, phantom, shift right=6ex, "\alpha_1"] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] \arrow[d] \arrow[r, phantom, shift right=6ex, "\alpha_2"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "|" marking] \arrow[d] \arrow[rrrr, phantom, shift right=6ex, "\alpha_3"] & & & \bullet \arrow[r, "|" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "|" marking] & & & & \bullet
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[d] \arrow[rrr, phantom, shift right=6ex, "\Phi(\alpha_1)"] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] \arrow[d] \arrow[r, phantom, shift right=6ex, "\Psi(\alpha_2)"] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "|" marking] \arrow[d] \arrow[rrrr, phantom, shift right=6ex, "\Gamma(\alpha_3)"] & & & \bullet \arrow[r, "|" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "|" marking] & & & & \bullet
\end{tikzcd}
$$

UNIQUENESS OF DECOMPOSITIONS:



RELATION TO LIMIT SKETCH:

Key Point: A VDC $\mathbb{D}$ has essentially unique decompositions of quaternary multicells into ternary, binary, and unary multicells, as below right, if and only if $\mathbb{D} \times -$ preserves the colimit below:

$$
\begin{tikzcd}
\mathbb{S}\mathsf{q}_3 & \mathbb{T}\mathsf{ight} \arrow[l] \arrow[r] & \mathbb{S}\mathsf{q}_1 \\
\mathbb{L}\mathsf{oose} \arrow[u, "t"] \arrow[dr, "s_1"'] & & \mathbb{L}\mathsf{oose} \arrow[u, "t"'] \arrow[dl, "s_2"] \\
& \mathbb{S}\mathsf{q}_2 &
\end{tikzcd}
\quad \overset{\mathsf{colim}}{\longmapsto} \quad
\begin{tikzcd}
x_0 \arrow[r, "\shortmid" marking] \arrow[d] \arrow[rrr, phantom, bend left=0, shift right=5ex, "\alpha_1"] & x_1 \arrow[r, "\shortmid" marking] & x_2 \arrow[r, "\shortmid" marking] & x_3 \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & x_4 \arrow[d] \\
y_0 \arrow[rrr, "\shortmid" marking] \arrow[d] \arrow[rrrr, phantom, shift right=5ex, "\alpha_3"] & & & y_1 \arrow[r, "\shortmid" marking] & y_2 \arrow[d] \\
z_0 \arrow[rrrr, "\shortmid" marking] & & & & z_1
\end{tikzcd}
$$

MAIN THEOREM

Theorem: Characterization of Exponentiable VDCs [EK26]

For a virtual double category $\mathbb{D}$, the following are equivalent:

1. The virtual double category $\mathbb{D}$ is exponentiable;
2. The multicells in $\mathbb{D}$ admit essentially unique decompositions;
3. The multicells in $\mathbb{D}$ admit essentially unique decompositions through binary, unary, and nullary multicells;
4. The exponential $\mathbb{S}\mathrm{pan}^{\mathbb{D}}$ exists [Ark26];
5. $\mathbb{D}$ is malleable, in the sense of [Ark26].

EXPONENTIABLE VDCS

Examples:

- All representable VDCs are exponentiable

$$
\begin{tikzcd}
x_0 \arrow[r, "\varphi_1"] \arrow[d, "a"'] & x_{m_1} \arrow[r, "\varphi_2"] & \cdots \arrow[r, "\varphi_r"] & x_{m_r} \arrow[d, "b"] \\
y_0 \arrow[rrr, "\psi"'] \arrow[urrr, phantom, "\alpha"] & & & y_1
\end{tikzcd}
=
\begin{tikzcd}
x_0 \arrow[r, "\varphi_1"] \arrow[d, equal] \arrow[dr, phantom, "\text{opcart}"] & x_{m_1} \arrow[r, "\varphi_2"] \arrow[d, equal] \arrow[dr, phantom, "\text{opcart}"] & \cdots \arrow[r, "\varphi_r"] \arrow[d, phantom, "\cdots"] \arrow[dr, phantom, "\text{opcart}"] & x_{m_r} \arrow[d, equal] \\
x_0 \arrow[r, "\odot(\varphi_1)"'] \arrow[d, "a"'] & x_{m_1} \arrow[r, "\odot(\varphi_2)"'] & \cdots \arrow[r, "\odot(\varphi_r)"'] & x_{m_r} \arrow[d, "b"] \\
y_0 \arrow[rrr, "\psi"'] \arrow[urrr, phantom, "\exists! \alpha^\flat"] & & & y_1
\end{tikzcd}
$$

EXPONENTIABLE VDCS

Examples:

- All representable VDCs are exponentiable
- For any category $\mathscr{E}$, $\mathbb{C}\mathrm{ospan}(\mathscr{E})$ is exponentiable

$$
\begin{tikzcd}
x_0 \arrow[r, "a_1"] \arrow[dd, "c_0"'] & y_1 \arrow[ddrr, "e_1"'] & x_1 \arrow[l, "b_1"'] \arrow[r, "a_2"] & \cdots & x_{n-1} \arrow[l, "b_{n-1}"'] \arrow[r, "a_n"] & y_n \arrow[ddll, "e_n"] & x_n \arrow[l, "b_n"'] \arrow[dd, "c_1"] \\
 & & & \cdots & & & \\
z_0 \arrow[rrr, "d_0"'] & & & w_1 & & & z_1 \arrow[lll, "d_1"]
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
x_0 \arrow[r] \arrow[d, "c_0"] & y_1 \arrow[dr, "e_1"] & \cdots \arrow[l] \arrow[r] & y_{k_1} \arrow[dl, "e_{k_1}"'] & x_{k_1} \arrow[l] \arrow[r] \arrow[d, "f_{k_1}"] & \cdots & x_{m_{n-1}} \arrow[l] \arrow[r] \arrow[d, "f_{m_{n-1}}"] & y_{m_{n-1}+1} \arrow[dr, "e_{m_{n-1}+1}"] & \cdots \arrow[l] \arrow[r] & y_{m_n} \arrow[dl, "e_{m_n}"'] & x_{m_n} \arrow[l] \arrow[d, "c_1"] \\
z_0 \arrow[rr, "d_0"'] \arrow[d, equal] & & w_1 \arrow[rr, equal] \arrow[drrr, equal] & & w_1 \arrow[r, equal] \arrow[dr, equal] & \cdots \arrow[r, equal] & w_1 \arrow[rr, equal] \arrow[dl, equal] & & w_1 \arrow[dlll, equal] & & z_1 \arrow[ll, "d_1"] \arrow[d, equal] \\
z_0 \arrow[rrrrr, "d_0"'] & & & & & w_1 & & & & & z_1 \arrow[lllll, "d_1"]
\end{tikzcd}
$$

EXPONENTIABLE VDCS

Examples:

- All representable VDCs are exponentiable
- For any category $\mathscr{E}$, $\mathbb{C}\text{ospan}(\mathscr{E})$ is exponentiable
- All functors below except $\mathbb{O}\text{b}_0$ preserve exponentiable objects, while $\mathbb{O}\text{b}_0$ detects exponentiability for VDCs with restrictions:

$$
\begin{tikzcd}[column sep=huge]
\mathbf{Multcat} \arrow[rr, hook, bend right, "B"'] \arrow[rr, phantom, "\perp"] & & \mathbf{Vbc} \arrow[ll, bend right, "\mathsf{Coll}_0"'] \arrow[rr, hook] & & \mathbf{Vdc} \arrow[ll, bend right, "\pi_0"'] \arrow[ll, bend left, "\mathbb{O}\mathrm{b}_0"] \arrow[ll, phantom, shift right=4, "\perp"] \arrow[ll, phantom, shift left=4, "\perp"]
\end{tikzcd}
$$

EXPONENTIABLE VDCS

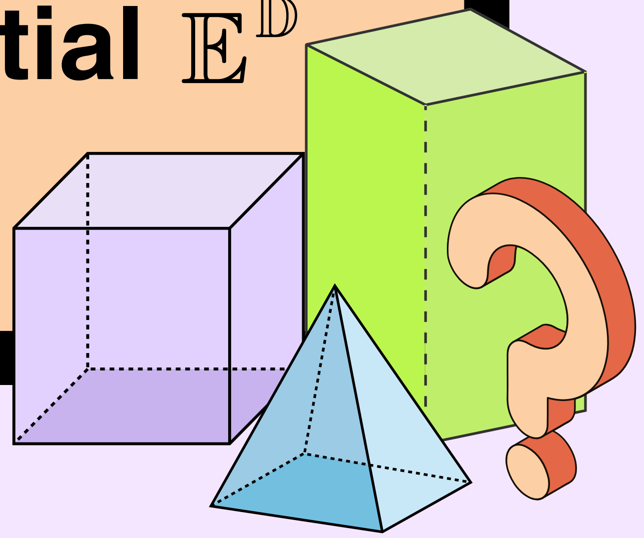
Examples:

- All representable VDCs are exponentiable
- For any category $\mathscr{E}$, $\mathbb{C}\mathrm{ospan}(\mathscr{E})$ is exponentiable
- All functors below except $\mathbb{O}\mathrm{b}_0$ preserve exponentiable objects, while $\mathbb{O}\mathrm{b}_0$ detects exponentiability for VDCs with restrictions
- VDCs with restrictions, lifts, and extensions are exponentiable

$$\begin{tikzcd}
\mathbf{Multcat} \arrow[r, bend right, "B"'] & \mathbf{Vbc} \arrow[l, bend right, "\mathsf{Coll}_0"'] \arrow[l, phantom, "\perp"] \arrow[r, hook] & \mathbf{Vdc} \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "\mathbb{O}\mathbf{b}_0"] \arrow[l, phantom, shift right=3ex, "\perp"] \arrow[l, phantom, shift left=3ex, "\perp"]
\end{tikzcd}$$

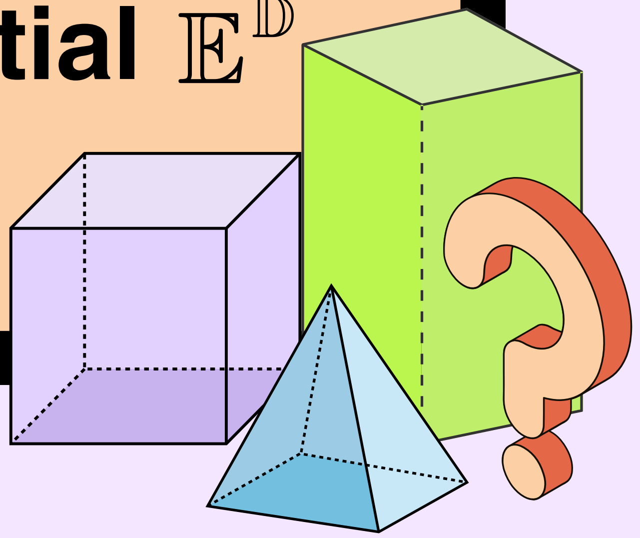
REPRESENTABILITY

Under what conditions is the exponential $\mathbb{E}^{\mathbb{D}}$ representable? What about $\mathbb{M}\mathrm{od}(\mathbb{E}^{\mathbb{D}})$?



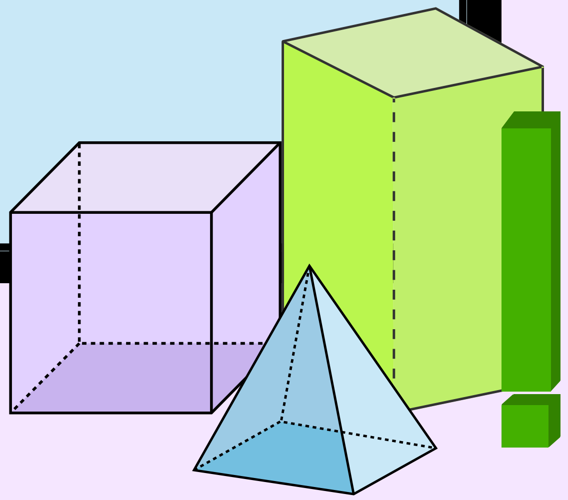
REPRESENTABILITY

Under what conditions is the exponential $\mathbb{E}^{\mathbb{D}}$ representable? What about $\mathbb{M}\mathrm{od}(\mathbb{E}^{\mathbb{D}})$?



Note: When $\mathbb{E}$ arises from a multicategory (resp. virtual bicategory), this is equivalent to asking when these are monoidal categories (resp. bicategories).

APPROXIMATING LOOSE ARROWS USING MULTICELLS TO DEFINE COMPOSITES



Key Idea: If we have a sequence of loose arrows in an exponential where the target has composites, we can attempt to define their composite as a local colimit:

$$(\Phi_1 \odot \cdots \odot \Phi_n)(\varphi : x \relbar\joinrel\mapstochar\joinrel\rightarrow y) := \operatorname*{colim}_{(\psi_1 \cdots \psi_n) \Rightarrow \varphi} \Phi_1(\psi_1) \odot \cdots \odot \Phi_n(\psi_n)$$

A DECOMPOSITION PROPERTY

Definition: Globular Decompositions

A virtual double category $\mathbb{A}$ has n-ary globular decompositions if any n-ary multicell:

$$\begin{tikzcd}
x_0 \arrow[r, "|" marking] & \cdots \arrow[r, "|" marking] & x_n \arrow[d] \\
y_0 \arrow[u, leftarrow] \arrow[rr, "|" marking] \arrow[urr, phantom, "\alpha"] & & y_1
\end{tikzcd}$$

admits a unique up to associativity decomposition through a globular multicell:

$$\begin{tikzcd}
x_0 \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\beta_1"] & x_1 \arrow[r, "|" marking] \arrow[d] & \cdots \arrow[r, "|" marking] & x_{n-1} \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\beta_n"] & x_n \arrow[d] \\
y_0 \arrow[r, "|" marking] \arrow[d, equal] & z_1 \arrow[r, "|" marking] & \cdots \arrow[r, "|" marking] \arrow[d, phantom, "\gamma"] & z_{n-1} \arrow[r, "|" marking] & y_1 \arrow[d, equal] \\
y_0 \arrow[rrrr, "|" marking] & & & & y_1
\end{tikzcd}$$

Examples for $\mathbb{A}$:

- Virtual bicategories
- Cospan VDCs
- $\mathrm{fc}(\mathbb{D})^{\text{op-t}}$ for any VDC $\mathbb{D}$

REPRESENTABILITY OF EXPONENTIALS

Theorem: Composites in Exponentials [EK26]

If $\mathbb{A}$ is a small exponentiable VDC with **globular decompositions** and $\mathbb{X}$ is a (weakly) **locally cocomplete** VDC with (weak) non-nullary composites, then $\mathbb{X}^{\mathbb{A}}$ has (weak) composites and $\mathbb{M}\mathrm{od}(\mathbb{X}^{\mathbb{A}})$ is (weakly) representable.

- Further, both are locally cocomplete and have restrictions if $\mathbb{X}$ does.

Note: When $\mathbb{X}$ is a virtual bicategory, no decomposition property is needed to obtain weak composites in $\mathbb{X}^{\mathbb{A}}$ and weak representability in $\mathbb{M}\mathrm{od}(\mathbb{X}^{\mathbb{A}})$.

REPRESENTABILITY OF EXPONENTIALS

Theorem: Composites in Exponentials [EK26]

If $\mathbb{A}$ is a small exponentiable VDC with **globular decompositions** and $\mathbb{X}$ is a (weakly) **locally cocomplete** VDC with (weak) non-nullary composites, then $\mathbb{X}^{\mathbb{A}}$ has (weak) composites and $\mathbb{M}\mathrm{od}(\mathbb{X}^{\mathbb{A}})$ is (weakly) representable.

- Further, both are locally cocomplete and have restrictions if $\mathbb{X}$ does.

Corollary:

For any VDC $\mathbb{D}$, we have an embedding into a locally cocomplete equipment:

$$\mathbb{D} \hookrightarrow \mathbb{F}_u(\mathbb{D}) \hookrightarrow \mathbb{V}\mathsf{df}(\mathsf{fc}(\mathbb{F}_u(\mathbb{D}))^{\text{op-t}}, \mathbb{S}\mathsf{pan}) \simeq \mathbb{V}\mathsf{df}_n(\mathsf{fc}(\mathbb{F}_u(\mathbb{D}))^{\text{op-t}}, \mathbb{P}\mathsf{rof})$$

KEY TAKEAWAYS

- Exponentiable VDCs are those admitting essentially unique multicell decompositions
- Composites in exponentials can be constructed approximating loose arrows by globular multicells

FUTURE DIRECTIONS

- Exponentiable normal VDCs
- Exponentiability in the context of lax functors and lax transformations as appearing in [LP24]
- Universal properties of presheaf embeddings for VDCs

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