EXPONENTIABLE VIRTUAL DOUBLE CATEGORIES

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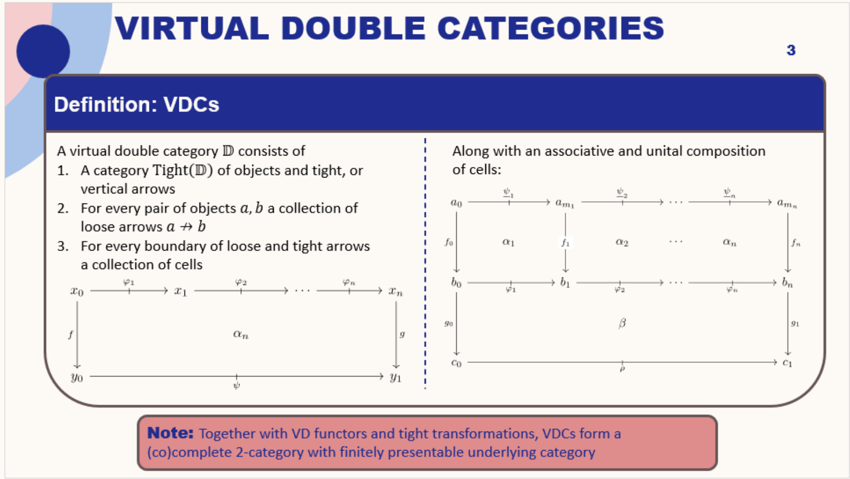
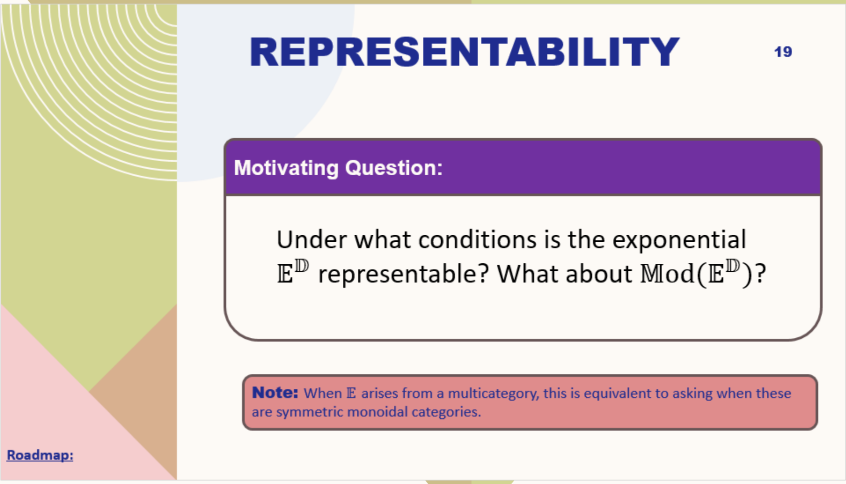
2025 Category Theory Octoberfest

ROADMAP

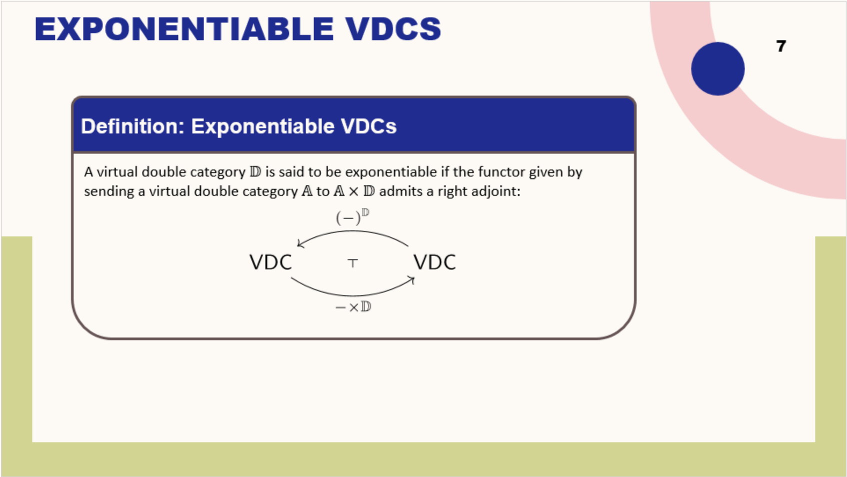
Intro to VDCs:



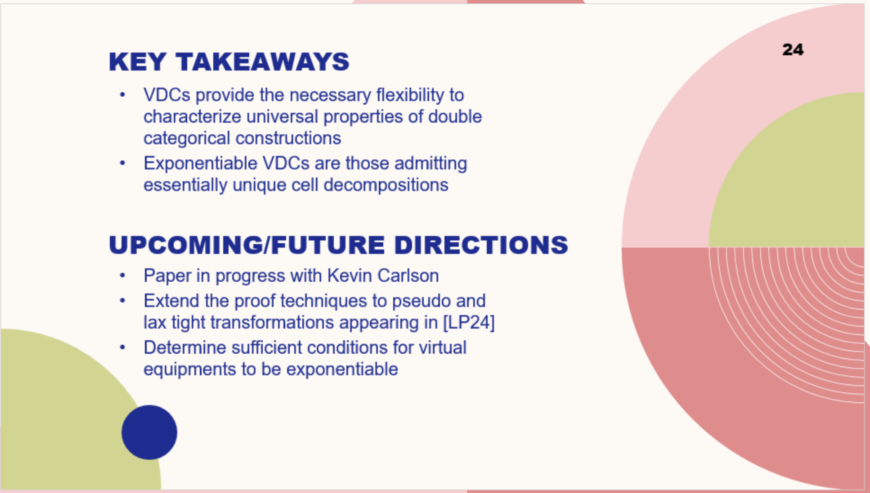
Representability:



Exponentiability:



Conclusions and Future Work:

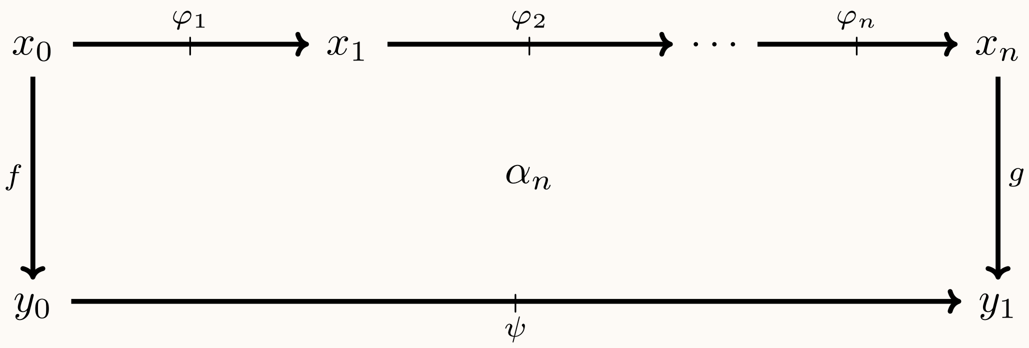


VIRTUAL DOUBLE CATEGORIES

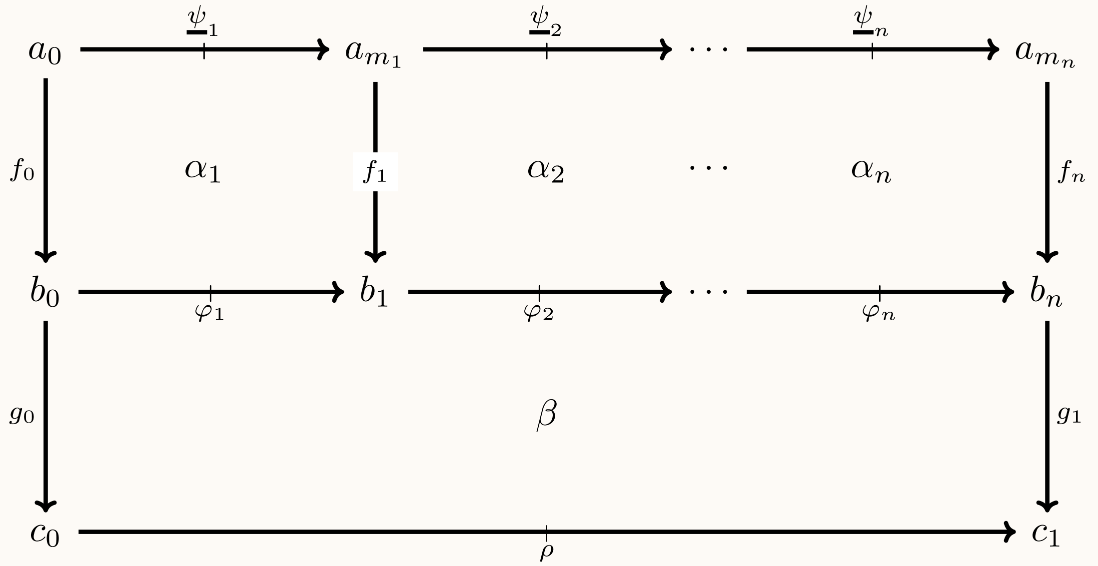
Definition: VDCs

A virtual double category $\mathbb{D}$ consists of

1. A category $\mathrm{Tight}(\mathbb{D})$ of objects and tight, or vertical arrows
2. For every pair of objects a, b a collection of loose arrows $a \nrightarrow b$
3. For every boundary of loose and tight arrows a collection of cells



Along with an associative and unital composition of cells:



Note: Together with VD functors and tight transformations, VDCs form a (co)complete 2-category with finitely presentable underlying category

MULTICATEGORIES AS VDCS

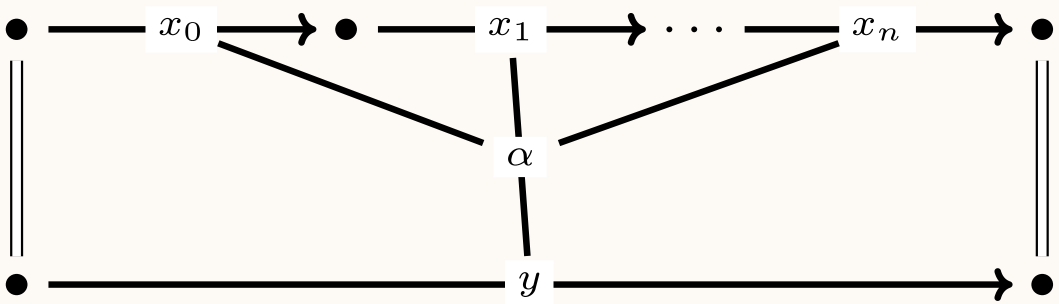
Example: Multicategories as VDCs

Multicategories embed fully-faithfully into VDCs:

$$\mathsf{MultCat} \underset{\Sigma}{\overset{\mathsf{Coll}}{\rightleftarrows}} \mathsf{Vdc} \quad (\bot)$$

as the VDCs with trivial underlying tight category.

That is, for a multicategory $\mathcal{M}$, multicells in $\Sigma\mathcal{M}$ are precisely multimorphisms in $\mathcal{M}$:



PSEUDO VS. VIRTUAL CONSTRUCTIONS

(Pseudo-)Double Categories:

- If $\mathcal{E}$ is a category with pushouts, we have a double category $\mathbb{C}\text{ospan}(\mathcal{E})$
- If $(\mathcal{V}, \otimes, I)$ is a monoidal category with finite coproducts that are preserved by $\otimes$, then we have a double category $\mathcal{V}\mathbb{M}\text{at}$
- If $\mathbb{D}$ is a double category with local reflexive co-equalizers, we have a double category $\mathbb{M}\text{od}(\mathbb{D})$ of modules in $\mathbb{D}$

Virtual Double Categories:

- For any category $\mathcal{E}$, we have a virtual double category $\mathbb{C}\text{ospan}(\mathcal{E})$
- For any virtual double category $\mathbb{D}$ we have a virtual double category $\mathbb{D}\mathbb{M}\text{at}$
- For any virtual double category $\mathbb{D}$, we have a unital virtual double category $\mathbb{M}\text{od}(\mathbb{D})$ of modules in $\mathbb{D}$

UNIVERSALITY OF VIRTUAL CONSTRUCTIONS

Universality of Constructions on Virtual Double Categories:

- For any category $\mathcal{E}$, the virtual double category $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is the free virtual equipment on $\mathcal{E}$ [DPP10]
- For any virtual double category $\mathbb{D}$ the virtual double category $\mathbb{D}\mathbb{M}\mathrm{at}$ is the free coproduct completion of $\mathbb{D}$ [Ark25]
- For any virtual double category $\mathbb{D}$, $\mathbb{M}\mathrm{od}(\mathbb{D})$ is the cofree normal completion of $\mathbb{D}$ [CS10]
- For any virtual double category $\mathbb{D}$, $\mathbb{D}\mathbb{P}\mathrm{rof} = \mathbb{M}\mathrm{od}(\mathbb{D}\mathbb{M}\mathrm{at})$ is the free collage cocompletion of $\mathbb{D}$ [Ark25]

EXPONENTIABLE VDCS

Definition: Exponentiable VDCs

A virtual double category $\mathbb{D}$ is said to be exponentiable if the functor given by sending a virtual double category $\mathbb{A}$ to $\mathbb{A} \times \mathbb{D}$ admits a right adjoint:

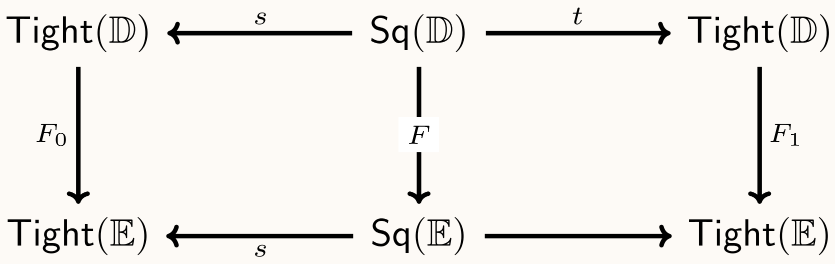
$$\mathrm{VDC} \underset{- \times \mathbb{D}}{\overset{(-)^{\mathbb{D}}}{\underset{\top}{\rightleftarrows}}} \mathrm{VDC}$$

EXPONENTIALS

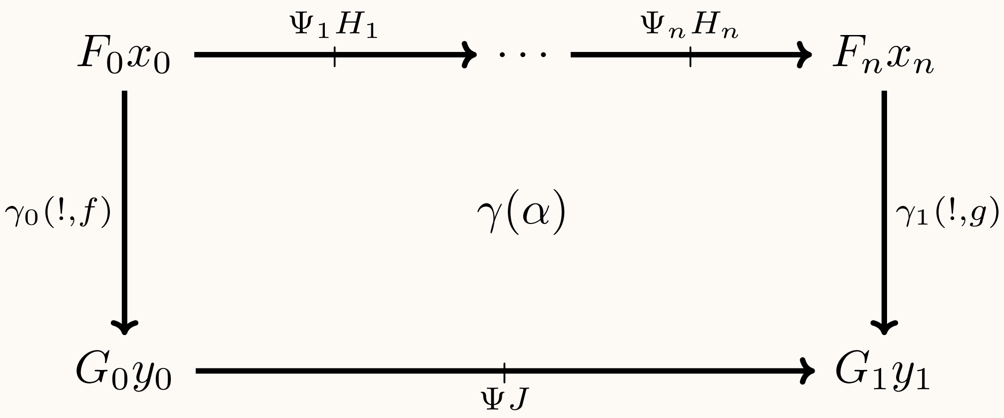
Explication: Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then it must consist of the following data:

- Objects are functors Tight($\mathbb{D}$) →Tight($\mathbb{E}$)
- Tight arrows are natural transformations
- Loose arrows are maps of spans



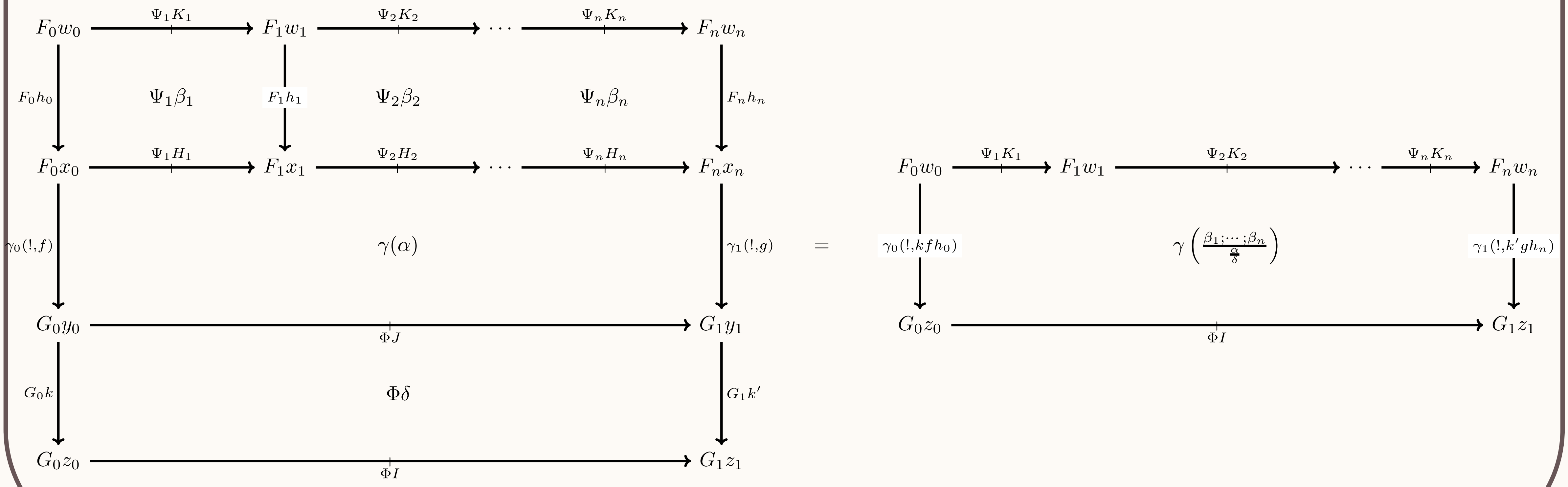
- n-ary multicells assign to each n-ary multicell in $\mathbb{D}$ an n-ary multicell in $\mathbb{E}$ with the following boundary:



… (cont. on next slide)

Explication: Exponential (cont.)

The assignment on multicells is subject to functoriality with respect to vertical pasting:



EXPONENTIABLE VDCS

Theorem: Characterization of Exponentiable VDCs

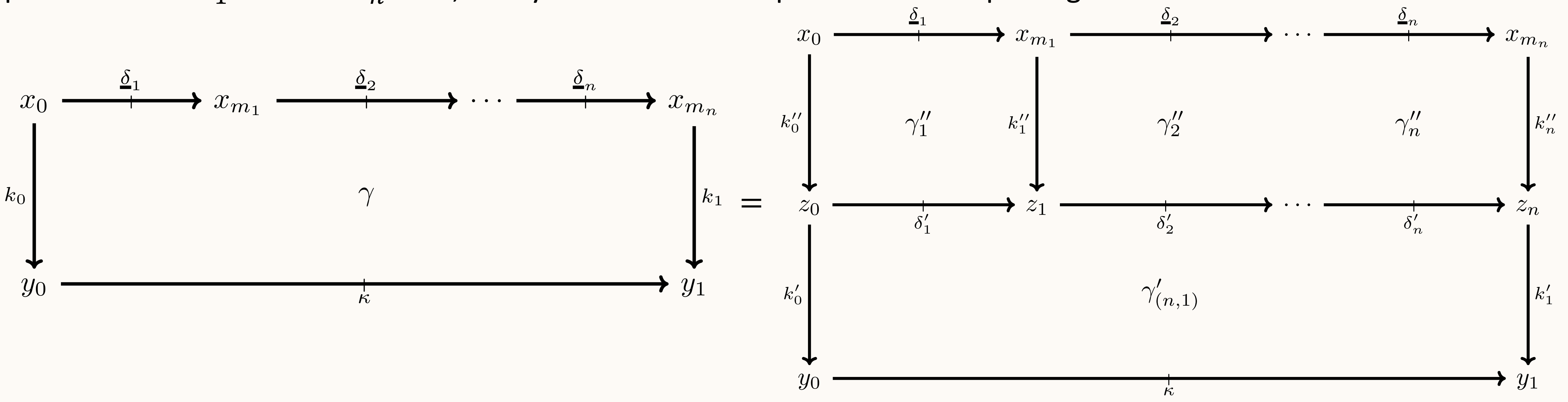
For a VDC $\mathbb{D}$, the following are equivalent:

1. The VDC $\mathbb{D}$ is exponentiable
2. The multicells in $\mathbb{D}$ admit essentially unique decompositions up to associativity
3. The multicells in $\mathbb{D}$ admit essentially unique decompositions up to associativity through binary, unary, and nullary multicells
4. The exponential $\mathbb{S}\mathrm{pan}^{\mathbb{D}}$ exists

EXPONENTIABLE VDCS

Explication: Characterization of Pro-representable VDCs

In terms of pasting diagrams, condition (2) says that a VDC $\mathbb{D}$ is exponentiable precisely when for any $N \geq 0$ and any partition $0 \leq m_1 \leq \cdots \leq m_n = N$, N-ary multicells decompose as vertical pastings:



and any two decompositions are equivalent up to associativity of pasting with cells in the center of the decomposition.

EXPONENTIABLE VDCS

Explication: Yoneda Characterization of Exponentiable VDCs

Condition (5) hints at, and follows from, the Yoneda theory of VDCs:

Yoneda Lemma for VDCs

For any VDC $\mathbb{D}$, there is a fully-faithful embedding of VDCs

$$\mathbb{D} \hookrightarrow \mathbb{S}\mathrm{pan}^{\mathbb{F}_s(\mathbb{D})^{op_t}}$$

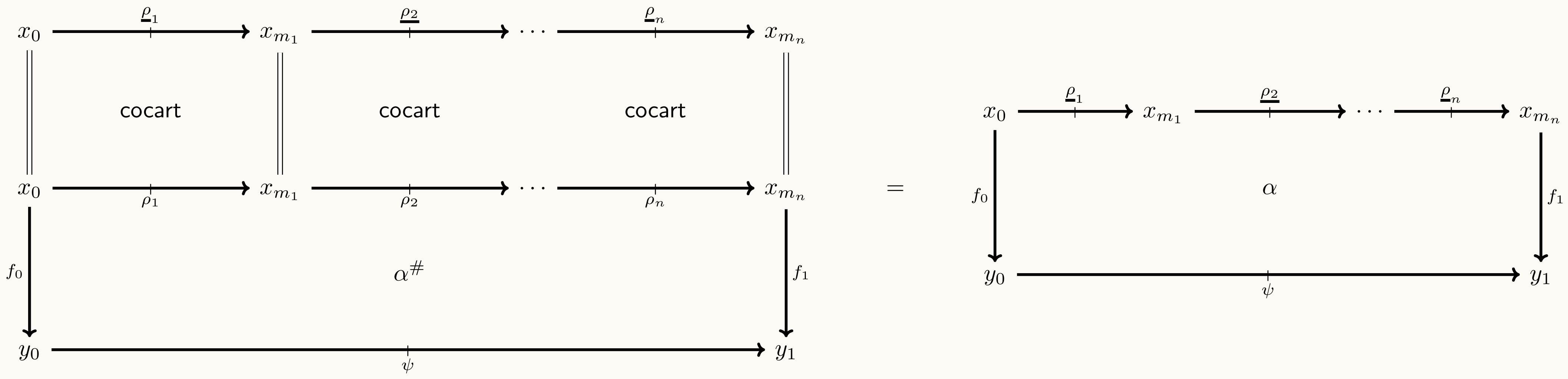
where $\mathbb{F}_s(\mathbb{D})$ is the free strict double category generated by $\mathbb{D}$.

REPRESENTABLE VDCS

Corollary: Representable $\Longrightarrow$ Exponentiable

Representable VDCs (i.e. pseudo-double categories) are exponentiable.

Proof Idea: Let $\mathbb{D}$ be a representable VDC. Then any cell admits a canonical decomposition



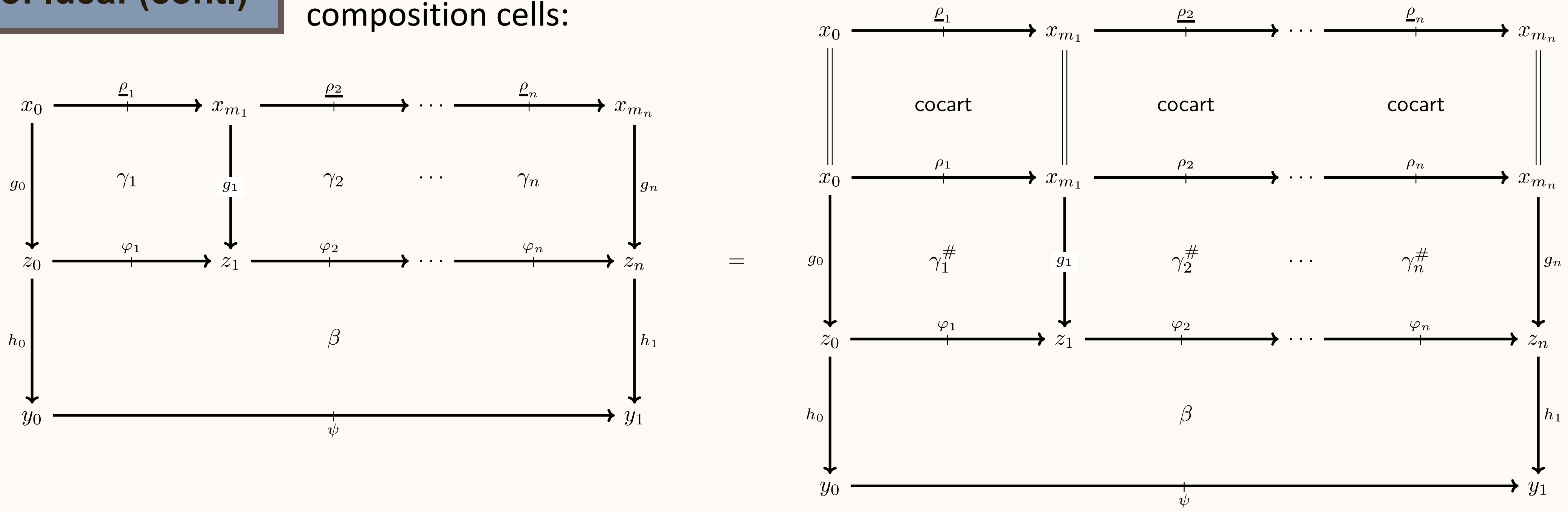
REPRESENTABLE VDCS

Corollary: Representable $\Longrightarrow$ Exponentiable

Representable VDCs (i.e. pseudo-double categories) are exponentiable.

Proof Idea: (cont.)

Any other decomposition below left can be canonically factored through the composition cells:

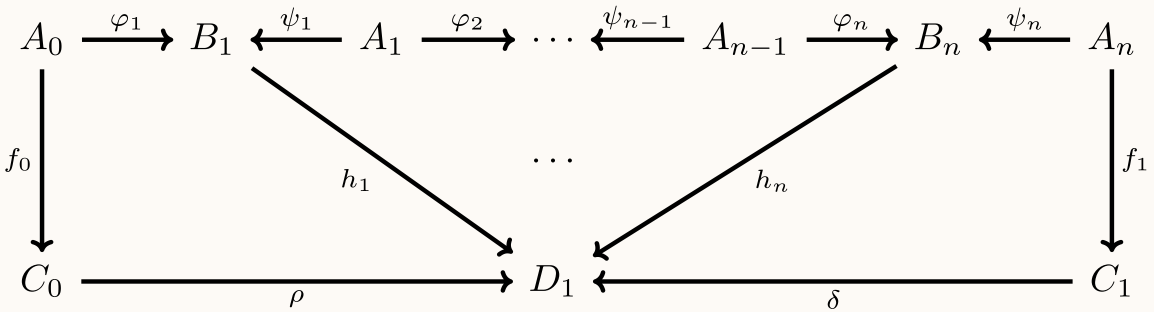


COSPANS ARE EXPONENTIABLE

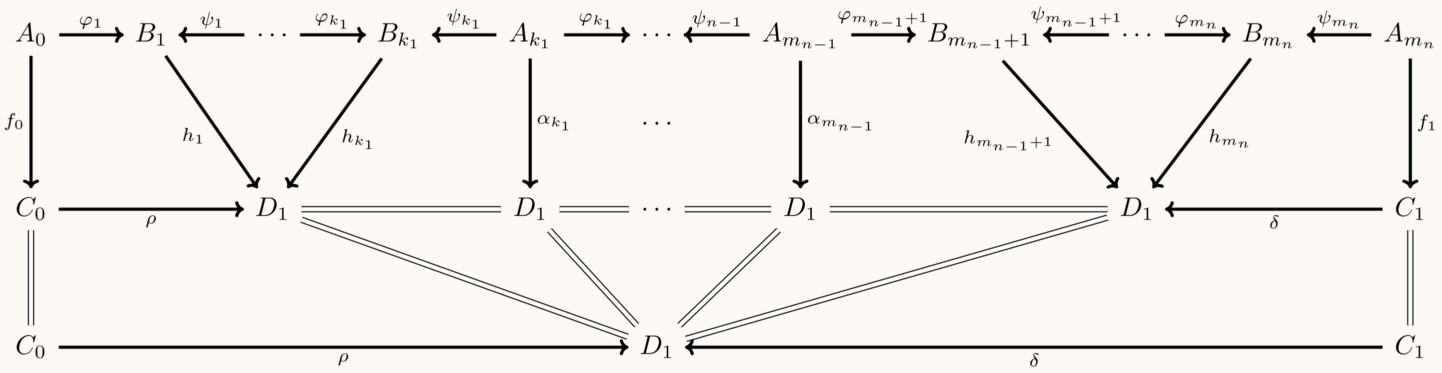
Proposition: Cospan VDCs are Exponentiable

For any category $\mathcal{E}$ the VDC $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is exponentiable, and it is representable if and only if $\mathcal{E}$ has finite pushouts.

Proof Idea: An arbitrary multicell:



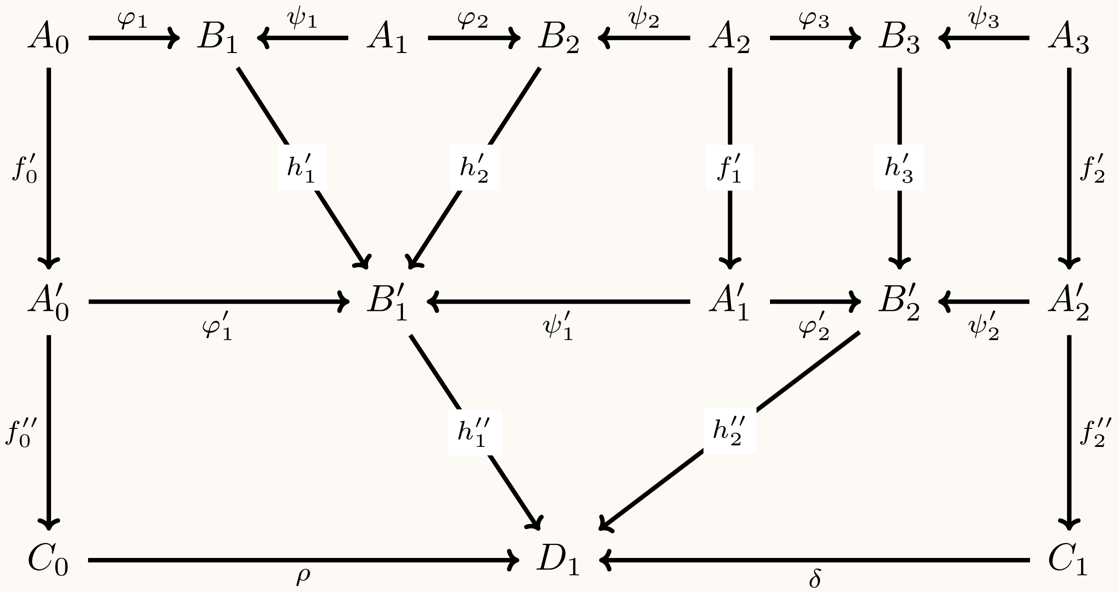
admits a canonical decomposition:



for any partition of n (the case where $k_1, k_n \geq 1$ is shown for simplicity).

COSPANS ARE EXPONENTIABLE

Proof Idea (cont): For uniqueness consider the case where $n = 3, k_1 = 2, k_2 = 1$ as an example. Then an arbitrary decomposition, below left, can be seen to be equivalent to the canonical decomposition via sliding cells:



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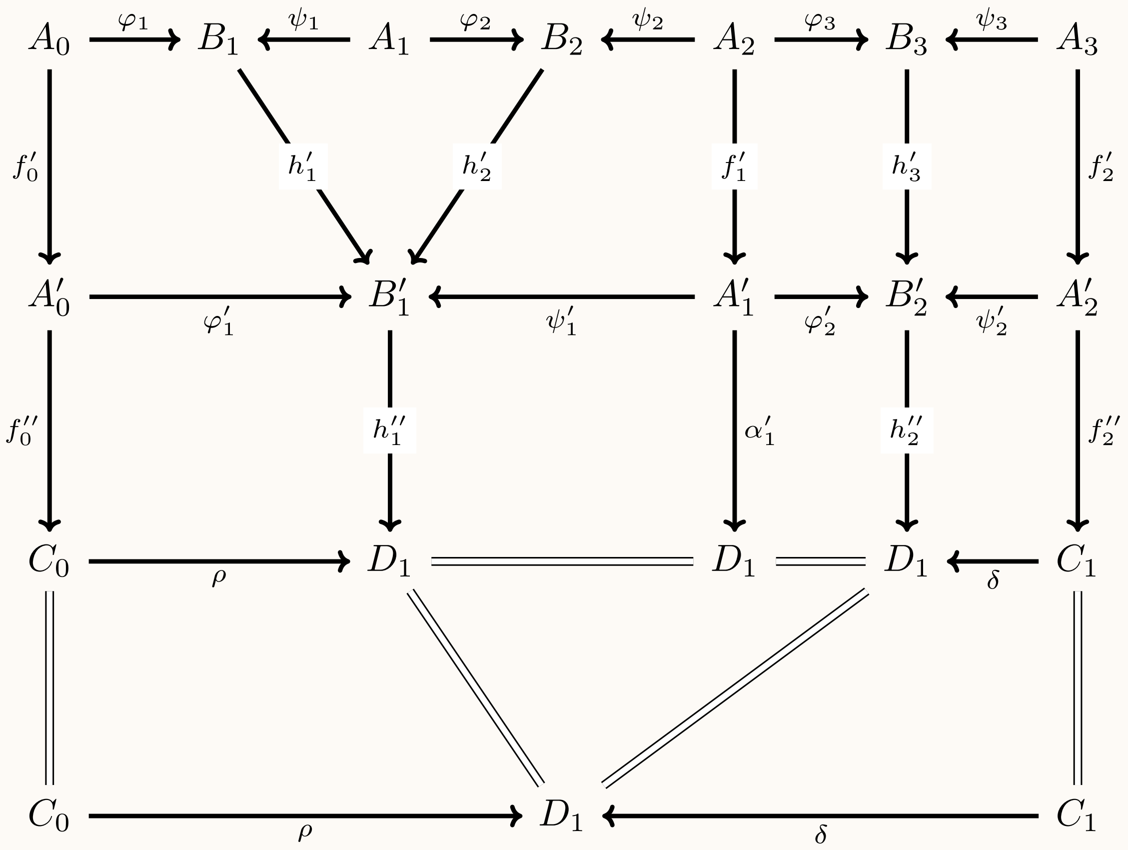
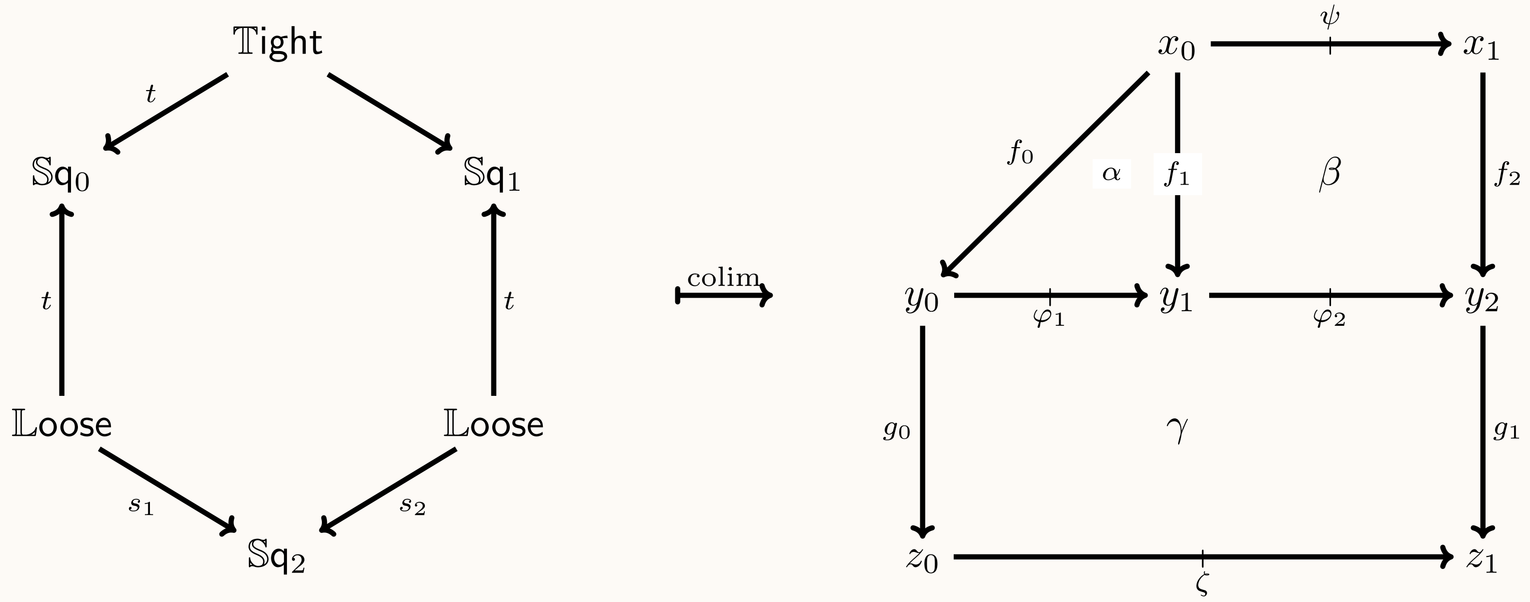


NON-EXPONENTIABLE VDC

Non-example: Non-unital Walking Loose Arrow

The VDC $\mathbb{L}\mathrm{oose}$ consisting of two objects 0 and 1 and a single loose arrow $0 \nrightarrow 1$ is not exponentiable. Consider the diagram and colimit $\mathbb{C}$ in VDC depicted below:



Then $\mathbb{C} \times \mathbb{L}\mathrm{oose}$ has two non-identity cells, while the VDC obtained by applying $- \times \mathbb{L}\mathrm{oose}$ to the diagram before taking the colimit only has one.

MODULES IN EXPONENTIAL VDC

Explication: Modules in Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then the VDC $\mathbb{M}\mathrm{od}(\mathbb{E}^{\mathbb{D}})$ consists of the following data:

- An object is a virtual double functor $F\colon \mathbb{D} \to \mathbb{E}$
- A tight arrow is a tight transformation between virtual double functors
- A loose arrow is a virtual double functor $F\colon \mathbb{L}\mathrm{oose}_{\mathrm{u}} \times \mathbb{D} \to \mathbb{E}$, where $\mathbb{L}\mathrm{oose}_{\mathrm{u}}$ is the unital walking loose arrow
- An n-multicell is a virtual double functor $\Gamma\colon \mathbb{S}\mathrm{q}_{\mathrm{n,u}} \times \mathbb{D} \to \mathbb{E}$, where $\mathbb{S}\mathrm{q}_{\mathrm{n,u}}$ is the unital walking n-multicell

REPRESENTABILITY

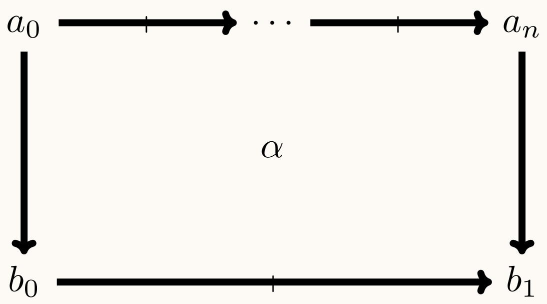
Motivating Question:

Under what conditions is the exponential $\mathbb{E}^{\mathbb{D}}$ representable? What about $\mathbb{M}\mathrm{od}(\mathbb{E}^{\mathbb{D}})$?

Note: When $\mathbb{E}$ arises from a multicategory, this is equivalent to asking when these are symmetric monoidal categories.

Definition: AFP Conditions (c.f. [Par13] for pseudo case)

A virtual double category $\mathbb{D}$ has AFP_n for $n \geq 2$ if any n-ary multicell α:



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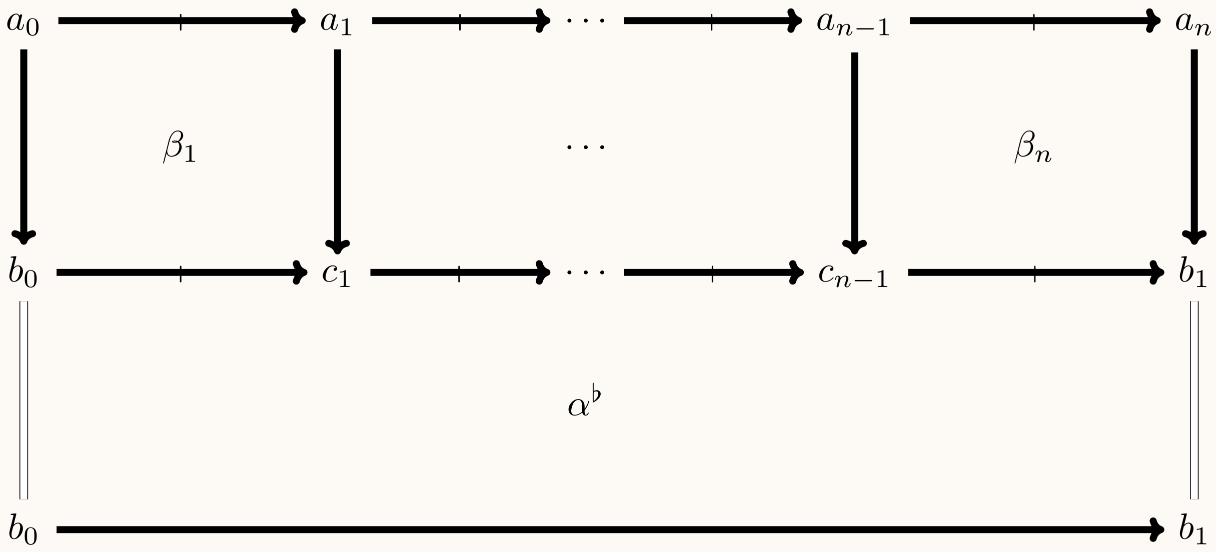
admits a unique up to associativity decomposition through a special n-ary multicell



REPRESENTABILITY

Example: VDCs Satisfying AFP Conditions

The following VDCs always satisfy the AFP conditions:

1. The cospan VDC, $\mathbb{C}\mathrm{ospan}(\mathcal{E})$, for any category $\mathcal{E}$
2. The VDC $\Sigma\mathcal{M}$ associated to an arbitrary multicategory $\mathcal{M}$
3. The VDC $\mathbb{L}\mathrm{oose}_u(\mathcal{B})$ for any bicategory $\mathcal{B}$

REPRESENTABILITY

Theorem: Non-nullary composites for Exponentials

If $\mathbb{A}$ is an exponentiable VDC satisfying the AFP conditions, and $\mathbb{X}$ is a (weakly) locally cocomplete VDC with (weak) non-nullary composites, then the exponential $\mathbb{X}^{\mathbb{A}}$ has (weak) non-nullary composites.

Corollary: Representability of $\mathbb{M}\mathrm{od}(\mathbb{E}^{\mathbb{D}})$

If $\mathbb{A}$ is an exponentiable VDC satisfying the AFP conditions, and $\mathbb{X}$ is a (weakly) locally cocomplete VDC with (weak) non-nullary composites, then $\mathbb{M}\mathrm{od}(\mathbb{E}^{\mathbb{D}})$ is (weakly) representable.

REPRESENTABILITY

Theorem: (Weak) composites for Exponentials into Tightly Discrete VDCs

If $\mathbb{A}$ is an exponentiable VDC and $\mathbb{X}$ is a locally cocomplete VDC with discrete tight category and weak composites, then the exponential $\mathbb{X}^{\mathbb{A}}$ has (weak) non-nullary composites.

Example: Colax Monoidal Convolution Structure

If $\mathbb{A}$ is an exponentiable VDC and $(\mathcal{C}, \otimes, I)$ is a cocomplete colax monoidal category with $\otimes$ preserving colimits in either variable, then $\mathcal{C}^{\mathsf{Sq}(\mathbb{A})}$ has a colax monoidal convolution structure induced by the weak representability of $\mathbb{L}\mathrm{oose}_u(B\mathcal{C})^{\mathbb{A}}$:

$$\int^{(p_1;\ldots;p_n)\in \mathsf{Sq}(\mathbb{A})^{\times_{\mathsf{Tight}(\mathbb{A})} n}} \mathbb{A}[p_1;\ldots;p_n,-] \star (F_1(p_1)\otimes\cdots\otimes F_n(p_n))$$

KEY TAKEAWAYS

- VDCs provide the necessary flexibility to characterize universal properties of double categorical constructions
- Exponentiable VDCs are those admitting essentially unique cell decompositions

UPCOMING/FUTURE DIRECTIONS

- Paper in progress with Kevin Carlson
- Extend the proof techniques to pseudo and lax tight transformations appearing in [LP24]
- Determine sufficient conditions for virtual equipments to be exponentiable

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