Intro to Paramaterized Infinity Categories

Introduction/Motivation

These notes are for a ~20 minute talk on Parameterized ∞-categories given at the start of the UIUC Fall 2025 Ambidexterity seminar. Primarily this note follows the structure and exposition in section I.2 of Bastiaan Cnossen's thesis[1] and the preliminaries section to Cnossen et al.'s paper on parameterized higher semiadditivity[2]. For more details on the foundations of this framework, along with detailed proofs, we refer to the sequence of papers by Martini and Wolf[3],[4],[5],[6],[7].

One of the primary motivations for developing such a theory comes from equivariant homotopy theory, where the topos in consideration is the ∞-category of presheaves on the orbit category of a finite group G. Another motivation for this approach comes from the observation that a continuous map of spaces $X \to Y$ induces a geometric morphism $\mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ of ∞-topoi. In this way, such a morphism allows us to consider $\mathsf{Sh}(X)$ as a $\mathsf{Sh}(Y)$-category which carries the information of the original sheaf topos on X along with topological properties of the continuous map inducing it.

Complete Segal Space and Sheaf Perspectives

Throughout $\mathcal{B}$ will be an ∞-topos, which is to say a *left exact and accessible localization* of a presheaf ∞-category $\mathsf{PSh}_{\mathcal{S}}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{S})$ for some small ∞-category $\mathcal{C}$, where $\mathcal{S}$ is the ∞-category of ∞-groupoids.

There are three main presentations of ∞-categories internal to the ∞-topos $\mathcal{B}$:

1. The simplicial perspective
2. The sheaf perspective
3. The fibered perspective

Simplicial Perspective

We write $\mathsf{Spine}[n] = [1] \cup_{[0]} \cdots \cup_{[0]} [1] \hookrightarrow [n]$ for the n-spine, which can be viewed as a simplicial ∞-groupoid $\mathsf{Spine}[n] : \mathbf{\Delta}^{op} \to \mathcal{S}$, and we write $\mathbb{E} = ([0] \sqcup [0]) \cup_{[1] \sqcup [1]} [3]$ for the walking equivalence. We will often use the algebraic morphism in the adjunction $(\mathsf{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}) : \mathcal{S}_{\Delta} \leftrightarrows \mathcal{B}_{\Delta}$ to identify simplicial ∞-groupoids, such as these, with simplicial objects in $\mathcal{B}$.

> **$\mathcal{B}$-category (Simplicial Definition)**
>
> A **$\mathcal{B}$-category** is a simplicial object $\mathsf{C} \in \mathcal{B}_{\Delta}$ that is internally local with respect to $\mathsf{Spine}[2] \hookrightarrow [2]$ (*Segal conditions*) and $\mathbb{E} \to [0]$ (*univalence*). We write $\mathsf{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_{\Delta}$ for the full subcategory spanned by $\mathcal{B}$-categories. A **$\mathcal{B}$-groupoid** is a simplicial object $G \in \mathcal{B}_{\Delta}$ which is internally local with respect to $[1] \to [0]$. We write $\mathsf{Grpd}(\mathcal{B}) \hookrightarrow \mathcal{B}_{\Delta}$ for the full subcategory spanned by $\mathcal{B}$-groupoids.

The $\mathcal{B}$-groupoids are precisely the essential image of the diagonal embedding $\mathcal{B} \hookrightarrow \mathcal{B}_{\Delta}$ so that $\mathcal{B} \simeq \mathsf{Grpd}(\mathcal{B})$. The statement that C is *internally local* with respect to a certain morphism $f : A \to B$ in $\mathcal{B}_{\Delta}$ means that the canonical map $\mathsf{C} \to [0]$ is *internally right orthogonal* to the morphisms, or equivalently that the induced map

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(B, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(A, \mathsf{C}) \times_{\underline{\mathsf{Fun}}_{\mathcal{B}}(A,[0])} \underline{\mathsf{Fun}}_{\mathcal{B}}(B, [0]) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(A, \mathsf{C})$$

is an equivalence. The inclusion $\mathsf{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_{\Delta}$ preserves filtered colimits and admits a left adjoint preserving finite products. In particular, $\mathsf{Cat}(\mathcal{B})$ is presentable and an exponential ideal in $\mathcal{B}_{\Delta}$, so in particular is cartesian closed.

We can define bifunctors

(*Functor ∞-category*)
$\mathsf{Fun}_{\mathcal{B}}(-,-) = \Gamma_{\mathcal{B}} \circ \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-) : \mathsf{Cat}(\mathcal{B})^{op} \times \mathsf{Cat}(\mathcal{B}) \to \mathsf{Cat}_{\infty}$
(*Powering*) $(-)^{(-)} = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{const}_{\mathcal{B}}(-), -) : \mathsf{Cat}_{\infty}^{op} \times \mathsf{Cat}(\mathcal{B}) \to \mathsf{Cat}(\mathcal{B})$
(*Tensoring*) $- \otimes - = \mathsf{const}_{\mathcal{B}}(-) \times - : \mathsf{Cat}_{\infty} \times \mathsf{Cat}(\mathcal{B}) \to \mathsf{Cat}(\mathcal{B})$
which fit into 2-variable adjunctions

$$\mathsf{Map}_{\mathsf{Cat}(\mathcal{B})}(- \otimes -, -) \simeq \mathsf{Map}_{\mathsf{Cat}_{\infty}}(-, \mathsf{Fun}_{\mathcal{B}}(-,-)) \simeq \mathsf{Map}_{\mathsf{Cat}(\mathcal{B})}(-, (-)^{(-)})$$

In particular, evaluating the left equivalence at the terminal ∞-category we have $\mathsf{Fun}_{\mathcal{B}}(-,-)^{\simeq} \simeq \mathsf{Map}_{\mathsf{Cat}(\mathcal{B})}(-,-)$, so that $\mathsf{Fun}_{\mathcal{B}}(-,-)$ gives rise to a

Cat_∞-enrichment of $\mathsf{Cat}(\mathcal{B})$, or in other words an $(\infty, 2)$-categorical enhancement.

Sheaf Perspective

Using our previously defined bifunctor, we have a natural fully-faithful embedding $\mathsf{Fun}_\mathcal{B}(\iota(-), -) : \mathsf{Cat}(\mathcal{B}) \hookrightarrow \mathsf{Fun}(\mathcal{B}^{op}, \mathsf{Cat}_\infty)$ which factors through an equivalence

$$\mathsf{Fun}_\mathcal{B}(\iota(-), -) : \mathsf{Cat}(\mathcal{B}) \xrightarrow{\simeq} \mathsf{Sh}_{\mathsf{Cat}_\infty}(\mathcal{B}) := \mathsf{Fun}^R(\mathcal{B}^{op}, \mathsf{Cat}_\infty)$$

Thus, $\mathcal{B}$-categories are equivalent to sheaves of ∞-categories $\mathcal{B}^{op} \to \mathsf{Cat}_\infty$. When $\mathcal{B} = \mathsf{Fun}(T, \mathcal{S})$ is a copre-sheaf ∞-topos, we have a natural equivalence

$$\mathsf{Fun}^R(\mathsf{Fun}(T, \mathcal{S})^{op}, \mathsf{Cat}_\infty) \simeq \mathsf{Fun}^L(\mathsf{Fun}(T, \mathcal{S}), \mathsf{Cat}_\infty^{op})^{op} \simeq \mathsf{Fun}(T^{op}, \mathsf{Cat}_\infty^{op})^{op} \simeq$$

Thus, the current perspective generalizes the notion of paramaterized ∞-categories. Additionally, via the un/straightening correspondence we obtain an embedding $\mathsf{Cat}(\mathcal{B}) \hookrightarrow \mathsf{Cart}(\mathcal{B})$.

For a $\mathcal{B}$-category C and $A \in \mathcal{B}$, we write $\mathsf{C}(A) := \mathsf{Fun}_\mathcal{B}(\iota(A), \mathsf{C})$ for the ∞-category of *local sections* over A, and write $s^* : \mathsf{C}(A) \to \mathsf{C}(B)$ for the restriction along a map $s : B \to A$ in $\mathcal{B}$. One can also interpret $\mathsf{C}(A)$ as the complete Segal space whose space of n-morphisms is given by the ∞-groupoid $\mathsf{Map}_\mathcal{B}(A, \mathsf{C}_n)$.

For a geometric morphism $f_* : \mathcal{B} \to \mathcal{A}$ with left adjoint f^*, $f_* : \mathsf{Sh}_{\mathsf{Cat}_\infty}(\mathcal{B}) \to \mathsf{Sh}_{\mathsf{Cat}_\infty}(\mathcal{A})$ is given by restriction along f^*, while $f^* : \mathsf{Sh}_{\mathsf{Cat}_\infty}(\mathcal{A}) \to \mathsf{Sh}_{\mathsf{Cat}_\infty}(\mathcal{B})$ is given by left Kan extension along $f^* : \mathcal{A} \to \mathcal{B}$. If f^* admits a further left adjoint $f_!$, then this is just precomposition with $f_!$.

Explication: Objects and Morphisms in $\mathcal{B}$-Categories

Using the two-variable adjunctions introduced previously, one obtains equivalences

$$\mathsf{C}^{[n]}(A)^{\simeq} \simeq \mathsf{Map}_{\mathsf{Cat}(\mathcal{B})}(A, \mathsf{C}^{[n]}) \simeq \mathsf{Map}_{\mathsf{Cat}(\mathcal{B})}([n] \otimes A, \mathsf{C}) \simeq \mathsf{Map}_{\mathsf{Cat}_\infty}([n], \mathsf{C}(A$$

for all $A \in \mathcal{B}$, $\mathsf{C} \in \mathsf{Cat}(\mathcal{B})$, and $n \geq 0$ (with the diagonal embedding $\iota : \mathcal{B} \hookrightarrow \mathsf{Cat}(\mathcal{B})$ left implicit). Combining this with a previous remark we see that

$$\mathsf{Map}_{\mathsf{Cat}(\mathcal{B})}(A, \mathsf{C}^{[n]}) \simeq \mathsf{Map}_{\mathcal{B}}(A, \mathsf{C}_n)$$

Let's see some examples in the sheaf perspective:

$\mathcal{B}$-groupoid

Every object B of $\mathcal{B}$ defines a $\mathcal{B}$-category $\underline{B}$ via the Yoneda embedding

$$\underline{B} := \mathsf{Map}_{\mathcal{B}}(-, B) : \mathcal{B}^{op} \to \mathcal{S} \hookrightarrow \mathsf{Cat}_\infty$$

The $\mathcal{B}$-categories of this form are called **$\mathcal{B}$-groupoids**.

The $\mathcal{B}$-category of $\mathcal{B}$-groupoids

Since $\mathcal{B}$ is a topos, the functor $\mathcal{B}^{op} \to \mathsf{Cat}_\infty$ associated to the cartesian fibration $\mathbf{ev}_1 : \mathcal{B}^{[1]} \to \mathcal{B}$ (i.e. $B \mapsto \mathcal{B}_{/B}$) preserves limits and thus defines a $\mathcal{B}$-category denoted by $\underline{\mathcal{S}}_{\mathcal{B}}$, and refer to it as the **$\mathcal{B}$-category of $\mathcal{B}$-groupoids**.

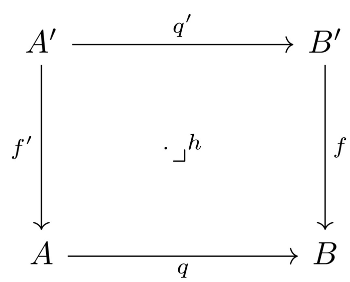
Internal (Co)limits

Let's now move to describing internal (co)limits in the parameterized setting.

$\mathcal{Q}$-colimits

Let $\mathcal{A}$ be an ∞-category and let $\mathcal{Q}$ be a class of morphisms in $\mathcal{A}$ closed under base change (i.e. base-changes of morphisms in $\mathcal{Q}$ exist and have output again in $\mathcal{Q}$). Given a functor $\mathcal{C} : \mathcal{A}^{op} \to \mathsf{Cat}_\infty$, we say that $\mathcal{C}$ **admits $\mathcal{Q}$-colimits** or is **$\mathcal{Q}$-cocomplete** if the following conditions are satisfied:

(1) For every $q : A \to B$ in $\mathcal{Q}$, the functor $q^* : \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $q_! : \mathcal{C}(A) \to \mathcal{C}(B)$
(2) For every pullback square in $\mathcal{A}$



with $q \in \mathcal{Q}$, the Beck-Chevalley transformation $BC_! : q'_! f'^* \Rightarrow f^* q_!$ of functors $\mathcal{C}(A) \to \mathcal{C}(B')$ is an equivalence.

Explication

If $B = 1$ is terminal, then the functor $q^* : \mathcal{C} \to \mathcal{C}(A)$ having a left adjoint $q_! : \mathcal{C}(A) \to \mathcal{C}$ is analogous to the diagonal $\Delta : \mathcal{E} \to \mathcal{E}^I$ having a left adjoint, which occurs precisely when $\mathcal{E}$ has I-shaped colimits. In general we can think of $q_!$ as a kind of left Kan extension. The Beck-Chevalley transformation then says that left Kan extensions formed in this way are preserved under base-change.

Dually, if q^* has a right adjoint $q_* : \mathcal{C}(A) \to \mathcal{C}(B)$ such that the corresponding Beck-Chevalley transformations are equivalences, then we say $\mathcal{C}$ **admits $\mathcal{Q}$-limits**.

Often we will add the assumption that $\mathcal{A} = \mathcal{B}$ is an ∞-topos, and that the class $\mathcal{Q}$ in $\mathcal{B}$ is **local** in the sense that a morphism $q : A \to B$ is in $\mathcal{Q}$ whenever there exists an effective epimorphism $\coprod_{i \in I} B_i \twoheadrightarrow B$ in $\mathcal{B}$ such that each of the base change maps $A \times_B B_i \to B_i$ is in $\mathcal{Q}$.

Preserving $\mathcal{Q}$-colimits

Let $\mathcal{C}, \mathcal{D} : \mathcal{A}^{op} \to \mathsf{Cat}_\infty$ be $\mathcal{Q}$-cocomplete. A natural transformation $F : \mathcal{C} \to \mathcal{D}$ is said to **preserve $\mathcal{Q}$-colimits** if for every morphism $q : A \to B$ in $\mathcal{Q}$ the Beck-Chevalley map $q_! F_A \to F_B q_!$ is an equivalence (we also say F is **$\mathcal{Q}$-cocontinuous**).

Example: Base Case

When $\mathcal{B} = \mathcal{S}$, global sections defines an equivalence $\Gamma : \mathsf{Cat}(\mathcal{S}) \xrightarrow{\simeq} \mathsf{Cat}_\infty$, with inverse given by sending $\mathcal{C}$ to $\mathsf{Fun}(-, \mathcal{C})$. For $\mathcal{Q} \subseteq \mathcal{S}$, a cat $\mathcal{C}$ then has $\mathcal{Q}$-colimits if and only if we can perform left Kan extension for space-indexed functors into $\mathcal{C}$ along morphisms in $\mathcal{Q}$.

In addition to *groupoid indexed colimits* defined in this fashion, we will also be interested in *fiberwise colimits*:

Fiberwise Colimits

> Let K be a (non-parameterized) ∞-category. We say that a $\mathcal{B}$-category $\mathcal{C}$ has **fiberwise K-shaped colimits** if the category $\mathcal{C}(A)$ has K-shaped colimits for every $A \in \mathcal{B}$ and the restriction functor $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$ preserves K-shaped colimits for each $f : A \to B$ in $\mathcal{B}$.

We say a functor of $\mathcal{B}$-categories with fiberwise K-shaped colimits **preserves fiberwise K-shaped colimits** if it preserves K-shaped colimits pointwise.

Combining these notions we will say a $\mathcal{B}$-category is **cocomplete** if it is $\mathcal{B}$-cocomplete in the parameterized sense and moeover *fiberwise cocomplete*. Analogously to the classical theory, if $\mathcal{C}$ is a cocomplete $\mathcal{B}$-category, then the inclusion of constant diagrams $\pi_A^* \mathcal{C} \to \underline{\mathsf{Fun}}(\mathcal{K}, \pi_A^* \mathcal{C})$ has a left adjoint for every $A \in \mathcal{B}$ and every small $\mathcal{B}_{/A}$-category $\mathcal{K}$ (c.f. Corollary 5.4.7 of[4-1]).

Symmetric Monoidal and Presentable $\mathcal{B}$-Categories

> **Definition: Symmetric Monoidal $\mathcal{B}$-category**
>
> A **symmetric monoidal $\mathcal{B}$-category** is a commutative monoid in the ∞-category $\mathsf{Cat}(\mathcal{B})$, or equivalently, a limit-preserving functor $\mathcal{C} : \mathcal{B}^{op} \to \mathsf{CMon}(\mathsf{Cat}_\infty)$.

For $B \in \mathcal{B}$ we denote the tensor product and monoidal unit of $\mathcal{C}(B)$ by $- \otimes_B -$ and I_B, respectively.

> **Definition: Presentable $\mathcal{B}$-Categories**
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> A $\mathcal{B}$-category $\mathcal{C} : \mathcal{B}^{op} \to \mathsf{Cat}_\infty$ is called **fiberwise presentable** if it factors (necessarily uniquely) through the subcategory $\mathsf{Pr}^L \subseteq \mathsf{Cat}_\infty$ of presentable ∞-categories and colimit preserving functors. We say $\mathcal{C}$ is **presentable** if it is fiberwise presentable and additionally satisfies the following two conditions:
>
> **(1)** *(Left Adjoints)* For all $f : A \to B$ in $\mathcal{B}$, the restriction $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$ has left adjoint $f_! : \mathcal{C}(A) \to \mathcal{C}(B)$
>
> **(2)** *(Left Base Change)* For every $B \xrightarrow{f} C \xleftarrow{g} A$, with pullback cospan $B \xleftarrow{g'} D \xrightarrow{f'} A$, the *Beck-Chevalley transformation* $g'_! f'^* \implies f^* g_!$ is

> an equivalence.

If $\mathcal{C}, \mathcal{D}$ are presentable $\mathcal{B}$-categories, we say $\mathcal{B}$-functor $F : \mathcal{B} \to \mathcal{C}$ *preserves (parameterized) colimits* if the following two properties are satisfied:

(1) For every $B \in \mathcal{B}$, the functor $F(B) : \mathcal{C}(B) \to \mathcal{D}(B)$ preserves small colimits;

(2) For every $f : A \to B$ in $\mathcal{B}$, the Beck-Chevalley transformation $f_! \circ F(A) \implies F(B) \circ f_!$ is an equivalence

Note that in the *left Base change* condition, passing to right adjoints gives that the other Beck-Chevalley transformation $g^* f_* \implies f'_* g'^*$ is also an equivalence by uniqueness of adjoints and the fact that $g'_! f'^* \dashv f'_* g'^*$ and $f^* g_! \dashv g^* f_*$. In fact, for a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{f^*} & B \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ C & \xrightarrow[k^*]{} & D \end{array}$$

with the morphisms fitting into adjoint triples $f_! \dashv f^* \dashv f_*$, etc., the Beck-Chevalley transformation

$$g_! k^* \xRightarrow{g_! k^* \eta} g_! k^* h^* h_! \simeq g_! g^* f^* h_! \xRightarrow{\epsilon f^* h_!} f^* h_!$$

is an equivalence if and only if the Beck-Chevalley transformation

$$h^* f_* \xRightarrow{\eta h^* f_*} k_* k^* h^* f_* \simeq k_* g^* f^* f_* \xRightarrow{k_* g^* \epsilon} k_* g^*$$

is an equivalence, since $g_! k^* \dashv k_* g^*$ and $f^* h_! \dashv h^* f_*$.

> ### $\mathcal{B}$-Parameterized Animae
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> The target functor $d_0 : \mathcal{B}^{\Delta^1} \to \mathcal{B}$ is a cartesian fibration, so by (HTT, Theorem 6.1.3.9) is classified by a limit-preserving functor
>
> $$\Omega_{\mathcal{B}} : \mathcal{B}^{op} \to \mathrm{Pr}^L, \quad B \mapsto \mathcal{B}_{/B}, \quad (f : A \to B) \mapsto (f^* : \mathcal{B}_{/B} \to \mathcal{B}_{/A})$$

> The pullback functors $f^* : \mathcal{B}_{/B} \to \mathcal{B}_{/A}$ have left adjoints $f \circ - : \mathcal{B}_{/A} \to \mathcal{B}_{/B}$ which satisfy the Beck-Chevalley condition, so $\Omega_{\mathcal{B}}$ is a presentable $\mathcal{B}$-category, called the **$\mathcal{B}$-category of $\mathcal{B}$-groupoids**.

As with un-parameterized presentable -categories, presentable $\mathcal{B}$-categories have a natural tensor product, characterized by the fact that it represents bi-cocontinuous $\mathcal{B}$-functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ (c.f.[6-1], Section 8.2). This gives $\mathsf{Pr}^L(\mathcal{B})$ the structure of a symmetric monoidal ∞-category $\mathsf{Pr}^L(\mathcal{B})^{\otimes}$ whose monoidal unit is $\Omega_{\mathcal{B}}$. In particular, the classical formula carries over to give

$$\mathcal{C} \otimes \mathcal{D} \simeq \underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathcal{C}^{op}, \mathcal{D})$$

The universal property satisfied by this parameterized monoidal structure can be expressed by the equivalence

$$\mathrm{Fun}^L_{\mathcal{B}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^L_{\mathcal{B}}(\mathcal{C}, \underline{\mathsf{Fun}}^L_{\mathcal{B}}(\mathcal{D}, \mathcal{E}))$$

which in particular implies that $\mathcal{C} \otimes - : \mathsf{Pr}^L_{\mathcal{B}} \to \mathsf{Pr}^L_{\mathcal{B}}$ preserves colimits.

> **Presentably Symmetric Monoidal $\mathcal{B}$-Category**
>
> A **presentably symmetric monoidal $\mathcal{B}$-category** is a commutative algebra object in the symmetric monoidal ∞-category $\mathsf{Pr}^L(\mathcal{B})^{\otimes}$.

Note that we have an embedding $\mathsf{Pr}^L(\mathcal{B})^{\otimes} \hookrightarrow \mathsf{Cat}(\mathcal{B})^{\times}$ so that a symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$ is presentably symmetric monoidal if and only if it is presentable, and the tensor product $\mathcal{B}$-functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bi-cocontinuous. This can be expressed by the following two non-parameterized conditions:

1. **(Fiberwise presentably symmetric monoidal)** For each $B \in \mathcal{B}$, the tensor product functor $- \otimes_B - : \mathcal{C}(B) \times \mathcal{C}(B) \to \mathcal{C}(B)$ is bi-cocontinuous
2. **(Left Projection Formula)** For each $f : A \to B$ in $\mathcal{B}$ and all objects $X \in \mathcal{C}(B)$ and $Y \in \mathcal{C}(A)$, the *exchange morphism*

   $$f_!(f^*(X) \otimes_A Y) \to f_!(f^*(X) \otimes_A f^* f_!(Y)) \to f_! f^*(X \otimes_B f_!(Y)) \to X \otimes_B f_!(Y)$$

   is an equivalence

Equivalently, this can be expressed as a limit-preserving functor $\mathcal{C} : \mathcal{B}^{op} \to \mathsf{CAlg}(\mathsf{Pr}^L)$ such that the symmetric monoidal restriction functors $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$ admit left adjoints that satisfy base change and satisfy the left projection formula.

Looking Forward

To continue the preliminaries needed for twisted ambidexterity, the next step is to understand the algebra and module theory associated to the symmetric monoidal ∞-category $\mathsf{Pr}^L(\mathcal{B})^{\otimes}$. Before diving into this material we will review some basic aspects of the theory of ∞-operads, and the theory algebras over algebraic patterns more generally.

References/Footnotes

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