Introduction to Semiadditivity

Introduction/Motivation

These notes are for a 30~40 minute talk on semi-additivity as appearing in Sections 2.1 and 2.2 of the paper *Ambidexterity and Height*[1], which was given as part of an Ambidexterity seminar at UIUC in Fall 2025. In the paper Carmeli, Schlank, and Yanovski use the theory of higher semi-additivity to abstract and generalize the notion of *height* appearing in chromatic homotopy theory. The v_n-self maps used in the definition of chromatic height are instead replaced by the *cardinalities* of certain π-finite spaces (to be discussed soon).

Further, the semi-additive height filtration introduced in the paper refines the inclusion $\mathsf{Sp}_{K(n)} \subseteq \mathsf{Sp}_{T(n)}$, which is now known to be strict for $n \geq 0$. We also pull some ideas from later work due to Cnossen et al. on Parameterized Semi-additivity[2], and earlier work on applications of ambidexterity to chromatic homotopy theory[3].

Let's begin by recalling the notation used in the paper:

Conventions:

- $\mathsf{Cat}_\infty^{\mathsf{st}} \subseteq \mathsf{Cat}_\infty$ will denote the sub-∞-category spanned by stable ∞-categories and exact functors. Similarly, $\mathsf{Pr}^L_{\mathsf{st}} \subseteq \mathsf{Pr}^L$ is the full-subcategory spanned by stable presentable ∞-categories.
- A space $A \in \mathcal{S}$ is
    - **m-finite** for $m \geq -2$, if $m = -2$ and A is contractible, or $m \geq -1$, the set $\pi_0 A$ is finite, and all fibers of the diagonal $\Delta : A \to A \times A$ are $(m-1)$-finite5
    - **π-finite** or **∞-finite**, if it is m-finite for some integer $m \geq -2$. For $-2 \leq m \leq \infty$ we write $\mathcal{S}_{m\mathsf{fin}} \subseteq \mathcal{S}$ for the full subcategory spanned by m-finite spaces.
    - **p-space**, for $p \in \mathbb{Z}$ a prime, if all the homotopy groups of A are p-groups.
- Given an ∞-category $\mathcal{C} \in \mathsf{Cat}_\infty$,
    - For every map of spaces $A \xrightarrow{q} B$, we write $q^* : \mathcal{C}^B \to \mathcal{C}^A$ for the pullback functor and $q_!$ and q_* for the left and right adjoints of q^* (i.e. given by left and right Kan extension, respectively), whenever they exist
    - Under the equivalence $\mathcal{S}_{/\mathsf{pt}} \simeq \mathcal{S}$, we will identify a space A with its map to the terminal object, so that above we would write A^* instead of q^* for $q : A \to \mathsf{pt}$, and similarly for the others
    - For each $X \in \mathcal{C}$, we write $X[A] := A_! A^* X$, and write $\nabla : X[A] \to X$ for the co-unit (called the **fold**). Similarly, we write $X^A := A_* A^* X$ and write $\Delta : X \to X^A$ for the unit (called the **diagonal**)

- Given a map of spaces $A \xrightarrow{q} B$, and $b \in B$, we write $q^{-1}(b)$ for the homotopy fiber of q over b. We say that:
  - An ∞-category $\mathcal{C}$ admits **q-limits** (resp. **q-colimits**) if it admits all limits (resp. colimits) of shape $q^{-1}(b)$ for all $b \in B$
  - A functor $F : \mathcal{C} \to \mathcal{D}$ **preserves q-limits** (resp. **preserves q-colimits**) if it preserves all limits (resp. colimits) of shape $q^{-1}(b)$ for all $b \in B$
- For every $-2 \le m \le \infty$
  - **m-finite (co)limits** refer to (co)limits indexed by an m-finite space
  - We write $\mathsf{Cat}_\infty^{m\mathsf{finColim}} \subseteq \mathsf{Cat}_\infty$ (resp. $\mathsf{Cat}^{m\mathsf{finLim}} \subseteq \mathsf{Cat}_\infty$) for the subcategory spanned by ∞-categories admitting m-finite colimits (resp. limits) and functors preserving them.
  - For $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}_\infty^{m\mathsf{finColim}}$ (resp. $\in \mathsf{Cat}_\infty^{m\mathsf{finLim}}$) We wrote $\mathsf{Fun}^{m-\mathsf{finL}}(\mathcal{C}, \mathcal{D})$ (resp. $\mathsf{Fun}^{m\mathsf{finR}}(\mathcal{C}, \mathcal{D})$) for the full subcategory of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ spanned by m-finite colimit (resp. limit) preserving functors
  - We write $\mathsf{Cat}_\infty^{\oplus\text{-}m} \subseteq \mathsf{Cat}_\infty$ for the subcategory spanned by the m-semiadditive ∞-categories and m-semiadditive (i.e. m-finite colimit preserving) functors.
  - Given an ∞-operad $\mathcal{O}$, we say $\mathcal{C} \in \mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ is compatible with $\mathcal{K}$-indexed colimits for some collection of ∞-categories $\mathcal{K}$ if the underlying ∞-category $\mathcal{C}$ admits $\mathcal{K}$-indexed colimits and every tensor operation $\otimes : \mathcal{C}^n \to \mathcal{C}$ of $\mathcal{O}$ preserves $\mathcal{K}$-indexed colimits in each variable
  - An **m-semiadditively $\mathcal{O}$-monoidal ∞-category** is an $\mathcal{O}$-monoidal m-semiadditive ∞-category which is compatible with m-finite colimits **Q. What does this mean?**
  - If $\mathcal{C}$ is a monoidal ∞-category and $\mathcal{D}$ is an ∞-category enriched in $\mathcal{C}$, we write $\mathsf{Hom}_{\mathcal{D}}^{\mathcal{C}}(X, Y)$ for the $\mathcal{C}$-mapping object of $X, Y \in \mathcal{D}$. When $\mathcal{C}$ is closed, we write $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ for $\mathsf{Hom}_{\mathcal{C}}^{\mathcal{C}}(X, Y)$. For every ∞-category $\mathcal{C}$ we write $\mathsf{Hom}_{\mathcal{C}}^{\mathcal{S}}(X, Y) = \mathsf{Map}_{\mathcal{C}}(X, Y)$.

The importance of m-finite maps and spaces lies in their use as indexing ∞-categories for diagrams that we are interested in comparing limits and colimits of. Specifically, the m in m-semiadditivity indicates the size of the π-finite spaces A for which we have norm maps

$$\mathsf{Nm}_A : \mathsf{colim}_A \Rightarrow \mathsf{lim}_A$$

which are equivalences.

Semiadditivity

Let's begin with the basic notion of ambidexterity in chromatic homotopy theory.

> **Ambidexterity of π-finite Maps**
>
> Let $\mathcal{C} \in \mathsf{Cat}_\infty$. A π-finite map $A \xrightarrow{q} B$ is called:

1. **weakly $\mathcal{C}$-ambidextrous** if it is an equivalence, or $\Delta_q : A \to A \times_B A$ is $\mathcal{C}$-ambidextrous
2. **$\mathcal{C}$-ambidextrous** if it is weakly $\mathcal{C}$-ambidextrous, $\mathcal{C}$ admits all q-limits and q-colimits, and the norm map $\mathrm{Nm}_q : q_! \to q_*$ is an equivalence.

A (-2)-finite map, i.e. an equivalence, is always $\mathcal{C}$-ambidextrous. If q is m-finite, then the diagonal

$$A \xrightarrow{\Delta_q} A \times_B A$$

is $(m-1)$-finite and the ambidexterity of Δ_q allows in turn the definition of Nm_q.

The property of being $\mathcal{C}$-ambidextrous is preserved by pullbacks and determined by its fibers. Since the fibers of the diagonal $A \to A \times A$ are path spaces of A, A is weakly $\mathcal{C}$-ambidextrous if and only if the path spaces of A are $\mathcal{C}$-ambidextrous. This begins the inductive construction since the path-space reduces from an m-finite space to an $(m-1)$-finite space.

Σ Prop: Characterization of $\mathcal{C}$-Ambidextrous Morphism

Let $\mathcal{C}$ be an ∞-category and let $A \xrightarrow{q} B$ be a π-finite map. The map is $\mathcal{C}$-ambidextrous if and only if the following hold:

1. q is weakly $\mathcal{C}$-ambidextrous
2. $\mathcal{C}$ admits all q-limits and q-colimits
3. Either q_* preserves all q-colimits or $q_!$ preserves all q-limits.

Proof Idea.
From the discussion above, we can assume wlog that $B = \mathrm{pt}$. The forward implication is immediate due to the norm equivalence, so it suffices to show that if **(1)-(3)** hold, then the norm map is an equivalence.
□ ***

Σ Prop: Closure of Ambidexterity under ∞-category constructions

Let $\mathcal{C}$ be an ∞-category and let A be a π-finite $\mathcal{C}$-ambidextrous space. The space A is also $\mathcal{D}$-ambidextrous for

- **(1)** $\mathcal{D} = \mathcal{C}^{op}$
- **(2)** $\mathcal{D} = \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ for $\mathcal{I}$ an ∞-category
- **(3)** $\mathcal{D} \subseteq \mathcal{C}$ containing the final object and closed under $\Omega^k_a A$-limits for all $a \in A$ and $k \geq 0$

- **(4)** $\mathcal{D} \subseteq \mathcal{C}$ containing the initial object and closed under $\Omega^k_a A$-colimits for all $a \in A$ and $k \geq 0$.

Note: Here $\Omega^k_a A$ is the k-fold based loop space at $a \in A$.

The first two properties are classical, while the last two are dual and follow from the inductive construction of norm maps.

Important

The main feature of ambidexterity is that it allows us to *integrate* families of morphisms in $\mathcal{C}$. That is, given a $\mathcal{C}$-ambidextrous map $A \xrightarrow{q} B$ and $X, Y \in \mathcal{C}^B$, we have a map

$$\int_q : \mathrm{Map}_{\mathcal{C}^A}(q^*X, q^*Y) \to \mathrm{Map}_{\mathcal{C}^B}(X, Y)$$

which can be given by the composite

$$\mathrm{Map}_{\mathcal{C}^A}(q^*X, q^*Y) \xrightarrow{q_!} \mathrm{Map}_{\mathcal{C}^B}(q_!q^*X, q_!q^*Y) \underset{\simeq}{\xleftarrow{-\circ \mathrm{Nm}_q}} \mathrm{Map}_{\mathcal{C}^B}(q_*q^*X, q_!q^*Y) \xrightarrow{\epsilon \circ - \circ \eta} \mathrm{Map}_{\mathcal{C}^B}$$

When $B = \mathrm{pt}$ we can think of an element of $\mathrm{Map}_{\mathcal{C}^A}(q^*X, q^*Y)$ as a map $A \xrightarrow{f} \mathrm{Map}_{\mathcal{C}}(X, Y)$, and $\int_A f \in \mathrm{Map}_{\mathcal{C}}(X, Y)$ as the sum of f over points of A. Explicitly, the identification of mapping spaces comes from the equivalences

$$\mathrm{Map}_{\mathcal{C}^A}(q^*X, q^*Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, q_*q^*Y) \simeq \lim_A \mathrm{Map}_{\mathcal{C}}(X, Y) = \mathrm{Map}_{\mathcal{C}}(X, Y)^A$$

Intuition for Induction: For a space A and a diagram $F : A \to \mathcal{C}$, to specify a norm map $\mathrm{Nm}_A : \mathrm{colim}_A F \to \lim_A F$ is to specify a homotopy coherently compatible collection of morphisms $\mathrm{Nm}^{a,b}_A : F(a) \to F(b)$, for $a, b \in A$. F always provides a *family* of candidates for these maps, $F_{a,b} : \mathrm{Map}_A(a, b) \to \mathrm{Map}_{\mathcal{C}}(F(a), F(b))$, but a-priori there is no coherent choice for them which can be made. But, if we can integrate over the spaces $\mathrm{Map}_A(a, b)$, we can just take

$$\mathrm{Nm}^{a,b}_A = \int_{A_{a,b}} F_{a,b}$$

When F is constant on some object X, then the Norm map is the same as a map of spaces $A \times A \to \mathrm{Map}_{\mathcal{C}}(X, X)$, where the above construction specializes to $\mathrm{Nm}^{a,b}_A = |\mathrm{Map}_A(a, b)|_X$.

Important

The initial claim comes from the natural equivalences

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_A F, \lim_A F) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}^A}(F, \underline{\lim_A F}) \simeq \mathrm{Map}_{\mathcal{C}^A}(F, \lim_A \overline{F}) \xrightarrow{\simeq} \mathrm{Map}_{(\mathcal{C}^A)^A}(\underline{F}, \overline{F})$$

where $\overline{F} : A \to \mathcal{C}^A$ is the whiskering of $F \circ \pi_1 : A \times A \to \mathcal{C}$, while $\underline{F} : A \to \mathcal{C}^A$ is the whiskering of $F \circ \pi_2 : A \times A \to \mathcal{C}$. Thus, we have a natural equivalence

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_A F, \lim_A F) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}^{A\times A}}(F \circ \pi_2, F \circ \pi_1)$$

Thus, the data of a map $\mathrm{colim}_A F \to \lim_A F$ is equivalent to the data of a map of simplicial sets $\alpha : [1] \times A \times A \to \mathcal{C}$ such that $\alpha_0 = F \circ \pi_2$ and $\alpha_1 = F \circ \pi_1$.

Further, in the case when $F = \underline{X}$ is the constant functor at some $X \in \mathcal{C}$. In this case, the data of a map $\mathrm{colim}_A X \to \lim_A X$ is equivalent to the data of a map of simplicial sets $\alpha_X : [1] \times A \times A \to \mathcal{C}$ such that $\alpha_X|_{\partial[1]\times A\times A} = \underline{X}$. A natural family of such transformations is then a map of simplicial sets $\alpha : [1] \times A \times A \times \mathcal{C} \to \mathcal{C}$ such that $\alpha|_{\partial[1]\times A\times A\times\mathcal{C}} = \pi_{\mathcal{C}}$.

More generally, if $\mathcal{C}$ has all A-shaped (co)limits, so that we have functors $\mathrm{colim}_A, \lim_A : \mathcal{C}^A \to \mathcal{C}$, then the natural equivalences we want are given by using adjoints and (co)continuity of the diagonal:

$$\mathrm{Nat}(\mathrm{colim}_A, \lim_A) \xrightarrow{\simeq} \mathrm{Nat}(\Delta_A \mathrm{colim}_A, \mathrm{id}_{\mathcal{C}^A}) \xrightarrow{\simeq} \mathrm{Nat}(\mathrm{colim}_A(\Delta_A \circ -), \mathrm{id}_{\mathcal{C}^A}) \xrightarrow{\simeq} \mathrm{Nat}(\Delta_A \circ -, \Delta_A)$$

Thus, a family of norms $\mathrm{Nm}_A : \mathrm{colim}_A \Rightarrow \lim_A$ is equivalent to a natural transformation $(\Delta_A \circ -) \Rightarrow \Delta_A : \mathcal{C}^A \to (\mathcal{C}^A)^A$.

Similarly, if $\mathcal{C}$ has all q-shaped (co)limits for $q : A \to B$, so that $q_! \dashv q^* \dashv q_*$ exist, then we have natural maps

$$\mathrm{Nat}(q_!, q_*) \xrightarrow{\simeq} \mathrm{Nat}(q^* q_!, \mathrm{id}_{\mathcal{C}^A}) \xrightarrow{-\circ \mathrm{BC}^L_{q^*,\pi_2^*}(\alpha)} \mathrm{Nat}((\pi_2)_! \pi_1^*, \mathrm{id}_{\mathcal{C}^A}) \xrightarrow{\simeq} \mathrm{Nat}(\pi_1^*, \pi_2^*)$$

where $\alpha : \pi_1^* q^* \Rightarrow \pi_2^* q^*$ is the natural equivalence coming from $\pi_1^* q^* \simeq (q\pi_1)^* = (q\pi_2)^* \simeq \pi_2^* q^*$, and the center map uses the mate calculus on this transformation to obtain $\mathrm{BC}^L_{q^*,\pi_2^*}(\alpha) : (\pi_2)_! \pi_1^* \Rightarrow q^* q_!$. If this Beck-Chevalley transformation is an equivalence, then it follows that the data of a norm $\mathrm{Nm}_q : q_! \Rightarrow q_*$ is equivalent to the data of a natural transformation $\pi_1^* \Rightarrow \pi_2^* : \mathcal{C}^A \to \mathcal{C}^{A\times_B A}$. Since the square in which α appears is a homotopy pullback square of ∞-groupoids, and hence we can assume without loss of generality that q is a Kan fibration, the square is *exact*, and hence it satisfies the **Beck-Chevalley condition** (c.f. Chapter 13 of[4]).

Inductive Approach: If A is an m-finite space, then the path spaces $\mathrm{Map}_A(a, b)$ are $(m-1)$-finite. Thus, if inductively we have invertible canonical norm maps Nm_B for all $(m-1)$-finite spaces B, then we obtain a canonical way to integrate $(m-1)$-finite families of morphisms, which allows us to define norm maps for all m-finite spaces. Whether all these new norm maps are isomorphisms is now a **property**, which if holding let's us continue the induction:

- **($m = -2$)** We define every ∞-category to be (-2)-semiadditive. Recall that the (-2)-finite spaces are the contractible ones, and the canonical norm map $\mathsf{Nm}_{\mathsf{pt}}$ is hence an equivalence, being equivalent to the identity transformation on $\mathrm{id}_{\mathcal{C}}$. This just says we can canonically sum a one point family of maps.
- **($m = -1$)** The only non-contractible (-1)-finite space is $A = \emptyset$. The associated norm map is the unique map

$$\mathsf{Nm}_{\emptyset} : 0_{\mathcal{C}} \to 1_{\mathcal{C}}$$

  from the initial object to the terminal object of $\mathcal{C}$, which always exists. Thus, $\mathcal{C}$ is (-1)-semiadditive if and only if it is *pointed*. This allows integration of empty families of morphisms, which is to say that for any $X, Y \in \mathcal{C}$, we get a canonical zero map given by the composition

$$X \to 1_{\mathcal{C}} \xleftarrow{\simeq} 0_{\mathcal{C}} \to Y$$

- **($m \geq 0$)** Let A be an m-finite space, and suppose $\mathcal{C}$ is $(m-1)$-semiadditive. Then in particular we have an equivalence $\mathsf{Nm}_{\Delta_A} : \Delta_{A,!} \overset{\simeq}{\Rightarrow} \Delta_{A,*} : \mathcal{C}^A \to \mathcal{C}^{A \times A}$, which corresponds to a wrong-way co-unit $\nu_{\Delta_A} : \Delta_A^* \Delta_{A,!} \Rightarrow \mathrm{id}$ and a wrong-way unit $\mu_{\Delta_A} : \mathrm{id} \Rightarrow \Delta_{A,*}\Delta_A^*$, so that we can define the map

$$\pi_1^* \xrightarrow{\eta} \Delta_{A,*}\Delta_A^*\pi_1^* \simeq \Delta_{A,*} \underset{\simeq}{\xleftarrow{\mathsf{Nm}_{\Delta_A}}} \Delta_{A,!} \simeq \Delta_{A,!}\Delta_A^*\pi_2^* \xrightarrow{\epsilon_{\Delta_A}} \pi_2^*$$

  which from the discussion preceding the induction is equivalent to a norm map $\mathsf{Nm}_q : q_! \Rightarrow q_*$, which is given by

$$q_! \xRightarrow{\eta \star q_!} q_* q^* q_! \underset{\simeq}{\xleftarrow{q_*\mathsf{BC}^L_{q^*,\pi_2^*}(\mathrm{id})}} q_*(\pi_2)_!\pi_1^* \xrightarrow{\eta} q_*(\pi_2)_!\Delta_{A,*}\Delta_A^*\pi_1^* \underset{\simeq}{\xleftarrow{\mathsf{Nm}_{\Delta_A}}} q_*(\pi_2)_!\Delta_{A,!}\Delta_A^*\pi_2^* \xrightarrow{q_*(\pi_2)_!\epsilon} q_*($$

  where the Beck-Chevalley transformation can be written as the composite

$$\mathsf{BC}^L_{q^*,\pi_2^*}(\alpha) : (\pi_2)_!\pi_1^* \xRightarrow{(\pi_2)_!\pi_1^*\star u_{q^*}} (\pi_2)_!\pi_1^* q^* q_! \xRightarrow{(\pi_2)_!\star\alpha\star q_!} (\pi_2)_!\pi_2^* q^* q_! \xRightarrow{c_{\pi_2}\star q^* q_!} q^* q_!$$

As a first example, in the $m = 0$ step A is equivalent to a set, so we can replace A by a set if necessary. Then $X : A \to \mathcal{C}$ is precisely a set of objects $(X_a)_{a \in A}$ in $\mathcal{C}$ indexed by those in A, and $\Delta_{A,!}X : A \times A \to \mathcal{C}$ is the matrix of objects $(X_{i,j})_{i,j \in A}$ with $X_{a,a} = X_a$ and $X_{a,b} = 0_{\mathcal{C}}$ when $a \neq b$, and similarly $\Delta_{A,*}X : A \times A \to \mathcal{C}$ is the matrix of objects $(X'_{i,j})_{i,j \in A}$ with $X'_{a,a} = X_a$ and $X'_{a,b} = 1_{\mathcal{C}}$ with $a \neq b$. On the other hand, $\pi_1^* X = (X_a)_{a,b \in A}$ and

$\pi_2^*X = (X_b)_{a,b\in A}$ are matrices with constant rows and constant columns, respectively. The composite

$$(X_a)_{a,b\in A} \to (X'_{a,b})_{a,b\in A} \xleftarrow{\simeq} (X_{a,b})_{a,b\in A} \to (X_b)_{a,b\in A}$$

is given precisely by the matrix of maps $f_{a,b} : X_a \to X_b$ with $f_{a,a} = \mathrm{id}_{X_a}$, while for $a \neq b$, $f_{a,b} : X_a \xrightarrow{!} 1_{\mathcal{C}} \xleftarrow{\simeq} 0_{\mathcal{C}} \xrightarrow{!} X_b$ is the unique composite through the zero object. The norm map is then the composite

$$\coprod_{a\in A} X_a \to \prod_{b\in A}\coprod_{a\in A} X_a \to \prod_{b\in A}\coprod_{a\in A} X'_{a,b} \xleftarrow{\simeq} \prod_{b\in A}\coprod_{a\in A} X_{a,b} \to \prod_{b\in A}\coprod_{a\in A} X_b \to \prod_{b\in A} X_b$$

As a second example, if we're doing the $m = 1$ step with A a connected 1-finite space, so that $A \cong BG$ for some finite group $G \cong \pi_1(A)$, then $\Delta_{A,!}X \simeq \coprod_{g\in G} X \simeq \prod_{g\in G} X \simeq \Delta_{A,*}X$. Write $A : A \to$ pt for the unique map to the point. Further, $A_!X = X_{hG}$ and $A^*X = X^{hG}$ for $X \in \mathcal{C}^{BG}$are the homotopy orbits and fixed points, respectively, while $A^*Y = Y$ for $Y \in \mathcal{C}$ is an object with trivial action, $\pi_1^*X = X$ is given $G \times G$-action with trivial right action component, and similarly for $\pi_2^*X = X$. Finally, if $Z \in \mathcal{C}^{BG\times BG}$ is a $G \times G$-$\mathcal{C}$ object, then $(\pi_1)_!Z = Z_{hG\times 1}$ is the G-space given by taking homotopy orbits with respect to the first factor, and $(\pi_1)_*Z = Z^{hG\times 1}$ is the G-space given by taking homotopy fixed points with respect to the first factor. Now, the composite

$$\pi_1^*X \xrightarrow{\Delta} \prod_{g\in G} X \xleftarrow{\simeq} \coprod_{g\in G} X \xrightarrow{\nabla} \pi_2^*X$$

is given by summing over orbits. Finally, the first map $X_{hG} \to (\underline{X_{hG}})^{hG}$ is given by sending the orbits of a G-space to the homotopy fixed points of the homotopy orbits with trivial action, and the last map $(\underline{X}_{hG})^{hG} \to X^{hG}$ is given by sending the the homotopy fixed points of the homotopy orbits of the original G-object viewed itself as a G-object with trivial action, to the homotopy fixed points of the underlying object. Thus, the resulting norm map is precisely the classical orbit map:

$$X_{hG} \xrightarrow{\simeq} (\underline{X_{hG}})^{hG} \xrightarrow{\Delta} \left(\left(\bigoplus_{g\in G} X\right)_{hG}\right)^{hG} \xrightarrow{\Delta} (\underline{X}_{hG})^{hG} \xrightarrow{\simeq} X^{hG}$$

given informally by $[x] \mapsto \sum_{g\in G} g \cdot x$.

Integrating the identity morphism yields the notion of $\mathcal{C}$-cardinality.

$\mathcal{C}$-cardinality

Let $\mathcal{C} \in \mathbf{Cat}_\infty$ and let $A \xrightarrow{q} B$ be a $\mathcal{C}$-ambidextrous map. We have a natural transformation $\mathrm{id}_{\mathcal{C}^B} \xrightarrow{|q|_{\mathcal{C}}} \mathrm{id}_{\mathcal{C}^B}$ given by the composition

$$\mathrm{id}_{\mathcal{C}^B} \xrightarrow{u_*} q_* q^* \underset{\simeq}{\xleftarrow{\mathrm{Nm}_q}} q_! q^* \xrightarrow{c_!} \mathrm{id}_{\mathcal{C}^B}$$

For a $\mathcal{C}$-ambidextrous space A, we write $\mathrm{id}_{\mathcal{C}} \xrightarrow{|A|_{\mathcal{C}}} \mathrm{id}_{\mathcal{C}}$ and call $|A|_{\mathcal{C}}$ the **$\mathcal{C}$-cardinality** of A
.

Note that for a given object $X \in \mathcal{C}$, $X \xrightarrow{|A|_X} X$ is exactly $\int_A \mathrm{id}_X$.

Motivating Example

Let $\mathcal{C}$ be a semiadditive ∞-category. For a finite set A, viewed as an 0-finite space, the operation $|A|_{\mathcal{C}}$ is simply the multiplication by the natural number which is the usual cardinality of A.

Note: For a $\mathcal{C}$-ambidextrous space A, the A-limits and A-colimits in $\mathcal{C}$ are canonically isomorphic, which implies the following:

Prop: Preservation of Limits and Colimits for Ambidextrous Spaces

Let $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}_\infty$, and let A be a $\mathcal{C}$- and $\mathcal{D}$-ambidextrous space. A functor $F : \mathcal{C} \to \mathcal{D}$ preserves all A-limits if and only if it preserves all A-colimits. Moreover, if F preserves all A-(co)limits, then $F(|A|_{\mathcal{C}}) \simeq |A|_{\mathcal{D}}$.

Using Fubini's theorem for iso-normed functors, we can obtain the following additivity result for cardinalities. In the current context Fubini's Theorem for iso-normed functors says that if $A \xrightarrow{p} B \xrightarrow{q} C$ are π-finite maps of π-finite spaces such that p and q are both $\mathcal{C}$-ambidextrous, then $\int_{qp}$ is homotopic to the composite

$$\mathsf{Map}_{\mathcal{C}^A}(p^*q^*X, p^*q^*Y) \xrightarrow{\int_p} \mathsf{Map}_{\mathcal{C}^B}(q^*X, q^*Y) \xrightarrow{\int_q} \mathsf{Map}_{\mathcal{C}^C}(X, Y)$$

Prop: Additivity of Cardinalities

Let $\mathcal{C} \in \mathsf{Cat}_\infty$ and $A \xrightarrow{q} B$ a map of spaces. If B and q are $\mathcal{C}$-ambidextrous, then A is $\mathcal{C}$-ambidextrous and for every $X \in \mathcal{C}$,

$$|A|_X = \int_B |q|_{B^*X}$$

Intuition: This says that the cardinality of the total space A is the *sum over B* of the cardinalities of the fibers A_b of q. To see how this is a consequence of Fubini we can re-write both sides using the integral notation to give

$$\int_A \mathrm{id}_{A^*X} \simeq \int_B \int_q \mathrm{id}_{q^*B^*X}$$

We can interpret this as saying

$$|A \times B|_{\mathcal{C}} = |A|_{\mathcal{C}} |B|_{\mathcal{C}} \in \mathsf{End}(\mathrm{id}_{\mathcal{C}})$$

and

$$|A|_{\mathcal{C}} = \coprod_{a \in \pi_0 A} |A_a|_{\mathcal{C}} \in \mathsf{End}(\mathrm{id}_{\mathcal{C}})$$

When $\mathcal{C}$ is monoidal and the tensor product preserves A-colimits in each variable, Lemma 3.3.4 of[3-1] implies that $|A|_X$ can be identified with $|A|_{\mathbb{1}} \otimes X$, where $\mathbb{1}$ is the monoidal unit. Additionally, if $R \in \mathsf{Alg}(\mathcal{C})$, then $|A|_R : R \to R$ can be identified with multiplication by the image of $|A|_{\mathbb{1}} \in \pi_0 \mathbb{1} := \pi_0 \mathsf{Map}(\mathbb{1}, \mathbb{1})$ under the unit map $\pi_0 \mathbb{1} \to \pi_0 R := \pi_0 \mathsf{Map}(\mathbb{1}, R)$, which we also denote by $|A|_R$.

Higher Commutative Monoids

We refer to an ∞-category as **m-semiadditive** if all m-finite spaces are ambidextrous. For $m = 0$ we recover the ordinary notion of a semiadditive ∞-category. Note that if $\mathcal{C} \subseteq \mathcal{D}$ is a full subcategory of an m-semiadditive ∞-category, then if $\mathcal{C}$ is either stable under m-finite colimits or m-finite limits, then it is stable under both, and it is m-semiadditive itself.

m-Commutative Monoids

Let $-2 \leq m < \infty$. For $\mathcal{C} \in \mathsf{Cat}_\infty^{m\mathsf{finLim}}$, the ∞-category of **m-commutative monoids** in $\mathcal{C}$ is given by

$$\mathsf{CMon}_m(\mathcal{C}) := \mathsf{Fun}^{m\mathsf{finR}}(\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})^{op}, \mathcal{C})$$

When $\mathcal{C} = \mathcal{S}$ we write $\mathsf{CMon}_m := \mathsf{CMon}_m(\mathcal{S})$, and refer to its objects as m-commutative monoids.

In the case $m = -2$, evaluating at pt, the unique object of $\mathsf{Span}(\mathcal{S}^{(-2)\mathsf{finColim}})$, gives an equivalence $\mathsf{CMon}_{-2}(\mathcal{C}) \simeq \mathcal{C}$.

Explication (CMon_m)

An object $X \in \mathsf{CMon}_m$ consists of an underlying space $X(\mathsf{pt})$, together with a collection of coherent operations for summation of m-finite families of points in it. Indeed, for $A \in \mathcal{S}_{m\mathsf{fin}}$, we have a canonical equivalence $X(A) \simeq X(\mathsf{pt})^A$. Given $A \to B$ in $\mathcal{S}_{m\mathsf{fin}}$, the image of $A = A \to B$ is the restriction $X(\mathrm{pt})^B \to X(\mathrm{pt})^A$, while the image of $B \leftarrow A = A$ encodes *integration along fibers* $X(\mathrm{pt})^A \to X(\mathrm{pt})^B$.

Question

How can we see the restriction and integration along fibers perspectives above?

Prop: Forgetful Functors between m-Commutative Monoids Cats

Let $-2 \leq m < \infty$ and let $\mathcal{C} \in \mathrm{Cat}_\infty^{(m+1)-\mathrm{finLim}}$. The restriction along the inclusion functor

$$\iota_m : \mathrm{Span}(\mathcal{S}_{m\mathrm{fin}}) \hookrightarrow \mathrm{Span}(\mathcal{S}_{(m+1)\mathrm{fin}})$$

induces a limit preserving functor

$$\iota_m^* : \mathrm{CMon}_{m+1}(\mathcal{C}) \to \mathrm{CMon}_m(\mathcal{C})$$

Proof.
It suffices to prove that ι_m preserves m-finite colimits. By the description of colimits in spans it suffices to prove that $\mathcal{S}_{m\mathrm{fin}} \hookrightarrow \mathcal{S}_{(m+1)\mathrm{fin}}$ is stable under m-finite colimits.
□

Question

How can we see that $\mathcal{S}_{m\mathrm{fin}}$ has m-finite colimits? If $A \xrightarrow{X} \mathcal{S}$ is an m-finite diagram, then

$$\mathrm{colim}_A X \simeq \mathrm{colim}_{A/X} * \simeq A/X$$

How do we know that A/X is also m-finite? We know that A is m-finite and that all fibers of the Kan fibration $A/X \to A$ are m-finite, so it is also an m-finite map. Do m-finite maps compose?

The following answers the above question:

Prop: m-finite Maps Compose

If $f : A \to B$ and $g : B \to C$ are m-finite, then so is their composite gf.

Proof.

Taking fibers, it suffices to show that if $f : A \to B$ is an m-finite map with B an m-finite space, then A is also m-finite. For each point $b \in B$, we have a homotopy fiber sequence $f^{-1}(b) \to A \to B$ where $f^{-1}(b)$ is also m-finite, by definition of m-finite maps. Thus, looking at the long exact sequence of homotopy groups for each $a \in f^{-1}(b)$, we see that A is also m-truncated, has finitely many path components, and has all homotopy groups begin finite, completing the proof.

□

We extend CMon_m to $m = \infty$ by defining for $\mathcal{C} \in \mathsf{Cat}_\infty^{\infty\mathsf{finLim}}$ the ∞-category

$$\mathsf{CMon}_\infty(\mathcal{C}) := \lim_m \mathsf{CMon}_m(\mathcal{C})$$

with limit computed in Cat_∞. This is equivalent to

$$\mathsf{Fun}^{\infty\mathsf{finR}}(\mathsf{Span}(\mathcal{S}_{\infty\mathsf{fin}})^{op}, \mathcal{C})$$

Consequently, when $\mathcal{C}$ is presentable, $\mathsf{CMon}_m(\mathcal{C})$ is presentable for all m, and $\mathsf{CMon}_\infty(\mathcal{C})$ can then be described as a colimit of $\mathsf{CMon}_m(\mathcal{C})$ in Pr^L:

Lemma: CMon_∞ as Colimit in Pr^L

For $\mathcal{C} \in \mathsf{Pr}^L$, the forgetful functors

$$\iota_m^* : \mathsf{CMon}_{m+1}(\mathcal{C}) \to \mathsf{CMon}_m(\mathcal{C})$$

admit left adjoints, and the colimit of the sequence

$$\mathcal{C} \simeq \mathsf{CMon}_{-2}(\mathcal{C}) \xrightarrow{\iota_{-1,!}} \mathsf{CMon}_{-1}(\mathcal{C}) \xrightarrow{\iota_{0,!}} \cdots$$

in Pr^L is $\mathsf{CMon}_\infty(\mathcal{C})$. In particular, $\mathsf{CMon}_\infty(\mathcal{C})$ is presentable.

The mapping spaces between two objects in an m-semiadditive ∞-category have a canonical m-commutative monoid structure.

Prop: Universality of $\mathsf{CMon}_m(-)$

Let $-2 \leq m \leq \infty$. For every $\mathcal{C} \in \mathsf{Cat}_\infty^{\oplus_m}$ and $\mathcal{D} \in \mathsf{Cat}_\infty^{m\mathsf{fin}}$, post-composition with evaluation at $\mathsf{pt} \in \mathcal{S}_{m\mathsf{fin}}$ induces an equivalence of ∞-categories

$$\mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}, \mathsf{CMon}_m(\mathcal{D})) \simeq \mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}, \mathcal{D})$$

As a consequence, for each m-semiadditive ∞-category we have a unique lift of the Yoneda embedding to a CMon_m-enriched Yoneda embedding:

Corollary: CMon_m-enriched Yoneda

Let $-2 \leq m \leq \infty$. For each $\mathcal{C} \in \mathsf{Cat}_\infty^{\oplus_m}$, there is a unique fully-faithful and m-semiadditive functor

$$よ^{\mathsf{CMon}_m} : \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{op}, \mathsf{CMon}_m)$$

whose composition with the forgetful functor $\mathsf{CMon}_m \to \mathcal{S}$ is the Yoneda embedding.

Here a functor between m-semiadditive ∞-categories is said to be **m-semiadditive** if it preserves m-finite limits.

Proof.
Taking $\mathcal{D} = \mathcal{S}$ in the universality of m-commutative monoids, we see that the ordinary Yoneda embedding

$$よ : \mathcal{C} \hookrightarrow \mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}^{op}, \mathcal{S}) \subseteq \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

lifts essentially uniquely to a fully-faithful m-finite limit preserving functor

$$よ^{\mathsf{CMon}_m} : \mathcal{C} \hookrightarrow \mathsf{Fun}^{m\mathsf{fin}}(\mathcal{C}^{op}, \mathsf{CMon}_m) \subseteq \mathsf{Fun}(\mathcal{C}^{op}, \mathsf{CMon}_m)$$

□

Currying we obtain a functor

$$\mathsf{Hom}^{\mathsf{CMon}_m}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{CMon}_m$$

lifting $\mathsf{Map}_{\mathcal{C}}(-,-)$, and hence giving each mapping space a canonical m-commutative monoid structure.

Examples

Before moving into more technical work, let's review some examples of m-semiadditive ∞-categories and the behaviour of cardinalities of m-finite spaces in them. We have the following universal example of an m-semiadditive ∞-category:

Universal Case

For $-2 \leq m < \infty$ the symmetric monoidal ∞-category of spans $\mathcal{C} = \mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})$ is the *universal m-semiadditive ∞-category*. For every $A \in \mathcal{S}_{m\mathsf{fin}}$, we have

$$|A|_{\mathsf{pt}} = (\mathsf{pt} \leftarrow A \rightarrow \mathsf{pt}) \in \pi_0 \mathsf{Map}_{\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})}(\mathsf{pt}, \mathsf{pt})$$

Note that $\pi_0 \mathsf{Map}_{\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}})}(\mathsf{pt}, \mathsf{pt})$ is the set of isomorphism classes of m-finite spaces with semiring structure given by

$$|A| + |B| = |A \sqcup B|, \quad |A| \cdot |B| = |A \times B|$$

Similarly, CMon_m is the universal *presentable m-semiadditive ∞-category*. The Yoneda embedding induces a fully-faithful m-semiadditive symmetric monoidal functor

$$\mathsf{Span}(\mathcal{S}_{m\mathsf{fin}}) \hookrightarrow \mathsf{CMon}_m$$

taking an m-finite space A to the free m-commutative monoid on A.

🕮 Homotopy Cardinality

For a π-finite space A, the **homotopy cardinality** of A is the rational number

$$|A|_0 := \sum_{a \in \pi_0(A)} \prod_{n \geq 1} |\pi_n(A, a)|^{(-1)^n} \in \mathbb{Q}_{\geq 0}$$

We say an ∞-category $\mathcal{C}$ is **semirational** if it is 0-semiadditive (i.e. 0-finite spaces are $\mathcal{C}$-ambidextrous, which are contractible, empty, and discrete spaces) and for each $n \in \mathbb{N}$, multiplication by n is invertible in $\mathcal{C}$ (e.g. $\mathsf{Sp}_{\mathbb{Q}}$ or $\mathbb{Q}\mathsf{Mod}$). Here multiplication by n on an object C is given by the cardinality $|\mathsf{pt}^{\sqcup n}|_C$, which is the composite

$$C \xrightarrow{\Delta} C^{\times n} \xleftarrow[\cong]{\mathsf{Nm}_{\mathsf{pt}^{\sqcup n}}} C^{\sqcup n} \xrightarrow{\nabla} C$$

A semirational ∞-category which admits all 1-finite colimits is automatically ∞-semiadditive, and for every π-finite space A, we have that its cardinality is its homotopy cardinality:

$$|A|_{\mathcal{C}} = |A|_0 \in \mathbb{Q}_{\geq 0} \subseteq \mathsf{End}(\mathrm{id}_{\mathcal{C}})$$

This comes from the fact that the cardinality is additive, and for every fiber sequence of π-finite spaces $F \to A \to B$ where B is connected, $|A| = |F||B|$.

In Chromatic homotopy theory we often come across examples of ∞-semiadditive ∞-categories of higher height. For a given prime p, and $0 \leq n < \infty$, let $K(n)$ be the Morava K-theory spectrum of height n at the prime p. We have that the localizations $\mathsf{Sp}_{K(n)}$ and $\mathsf{Sp}_{T(n)}$ are ∞-semiadditive. For $n = 0$, $\mathsf{Sp}_{K(0)} \simeq \mathsf{Sp}_{T(0)} \simeq \mathsf{Sp}_{\mathbb{Q}}$, and the cardinalities recover the homotopy cardinality. Similarly, since $\mathsf{Sp}_{K(n)}$ is p-local for all n, if A is a π-finite space whose homotopy groups have cardinality prime to p, then the $K(n)$-local cardinality of A coincides with the homotopy cardinality for all n by the previous example. However, this does not hold in general for π-finite spaces whose cardinality is not prime to p.

To study the $K(n)$-local cardinalities of π-finite spaces, it is useful to consider their image in Morava E-theory. For $n \geq 1$, let E_n be the Morava E-theory associated with some formal group of height n over $\overline{\mathbb{F}}_p$, viewed as an object of $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$. In particular, we have a (non-canonical) isomorphism

$$\pi_* E_n \cong \mathbb{W}(\overline{\mathbb{F}}_p)[[u_1, \ldots, u_{n-1}]][u^{\pm 1}], \quad |u_i| = 0, \quad |u| = 2$$

Chromatic Cardinality

The ∞-category $\Theta_n := \mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$ is ∞-semiadditive by Theorem 5.3.1 in6, and hence we can consider cardinalities of π-finite spaces in $\pi_0 E_n$. The **p-typical height n cardinality** of a π-finite space A is defined to be

$$|A|_n := |A|_{\Theta_n} \in \pi_0 E_n$$

For $n = 0$ we can identify $\overline{\mathbb{Q}}$ with $\pi_0 E_0$, and so can recover the homotopy cardinality. For $n > 0$, let $\widehat{L}_p A := \mathrm{Map}(B\mathbb{Z}_p, A)$ be the p-adic free loop space of A. It turns out that $|A|_n \in \pi_0 E_n$ belongs to the subring $\mathbb{Z}_{(p)} \subseteq \pi_0 E_n$ and satisfies $|A|_n = |\widehat{L}_p A|_{n-1}$. Applying this inductively we see that

$$|A|_n = |\mathrm{Map}(B\mathbb{Z}_p^n, A)|_0 \in \mathbb{Z}_{(p)}$$

for the p-typical height n cardinality in terms of the homotopy cardinality. If A is a p-space, then $\widehat{L}_p A \simeq LA := \mathrm{Map}(S^1, A)$ coincides with the ordinary loop space.

Question

How can we show that $\widehat{L}_p A \simeq LA$ when A is a p-space? Hint: First consider the universal examples $K(\mathbb{Z}/p, n)$.

The following gives another family of examples of higher semiadditive ∞-categories:

Prop: $\mathrm{Cat}_\infty^{m\mathrm{finColim}}$ is m-semiadditive

For every $-2 \leq m \leq \infty$ the ∞-category $\mathrm{Cat}_\infty^{m\mathrm{finColim}}$ is m-semiadditive.

Categorical Cardinality

Let $-2 \leq m \leq \infty$ and let $\mathcal{C} \in \mathsf{Cat}_\infty^{m\mathsf{finColim}}$. For every m-finite space A, the m-semiadditive structure of $\mathsf{Cat}_\infty^{m\mathsf{finColim}}$ gives rise to a functor $|A|_\mathcal{C} : \mathcal{C} \to \mathcal{C}$. When $m < \infty$, $|A|_\mathcal{C} \simeq \mathsf{colim}_A \Delta_{(-)}$ is given by taking the constant colimit on A. Since $\mathsf{Cat}_\infty^{\infty\mathsf{finColim}} \to \mathsf{Cat}_\infty^{m\mathsf{finColim}}$ preserves limits, and hence is m-semiadditive, the same claim holds for $m = \infty$.

Conversely, the m-semiadditive structure on $\mathsf{Cat}_\infty^{m\mathsf{finLim}}$ is given by taking limits of constant diagrams.

(co)Cartesian m-commutative Monoid Structure

For $\mathcal{C} \in \mathsf{Cat}_\infty^{m\mathsf{finColim}}$, since $\mathcal{S}_{m\mathsf{fin}}$ is freely generated from a point under m-finite colimits, we have

$$\mathsf{Map}^{\mathsf{mfinL}}(\mathcal{S}_{m\mathsf{fin}}, \mathcal{C}) \simeq \mathsf{Map}(\mathsf{pt}, \mathcal{C}) \simeq \mathcal{C}^{\simeq}$$

and the resulting m-commutative monoid structure on $\mathcal{C}^{\simeq}$ is referred to as the **cocartesian structure**. Dually, for $\mathcal{C} \in \mathsf{Cat}_\infty^{m\mathsf{finLim}}$, we have

$$\mathsf{Map}^{m\mathsf{finR}}(\mathcal{S}_{m\mathsf{fin}}^{op}, \mathcal{C}) \simeq \mathsf{Map}(\mathsf{pt}, \mathcal{C}) \simeq \mathcal{C}^{\simeq}$$

and the resulting m-commutative monoid structure on $\mathcal{C}^{\simeq}$ is referred to as the **cartesian structure**.

The full subcategory $\mathsf{Cat}_\infty^{\oplus_m} \subseteq \mathsf{Cat}_\infty^{m\mathsf{finColim}}, \mathsf{Cat}_\infty^{m\mathsf{finLim}}$ is closed under colimits, and in particular is m-semiadditive, since the inclusion admits the right adjoint $\mathsf{CMon}_m(\mathcal{D})$.

Extra Examples of Ambidexterity

Similarity between Ambidexterity and Traces

Recall that for a symmetric monoidal ∞-category $(\mathcal{C}, \otimes, 1)$ with subcategory $\mathcal{C}^\diamond \subseteq \mathcal{C}$ spanned by dualizable objects, every $X \in \mathcal{C}^\diamond$ admits a **trace** or **Euler characteristic** given by the composite

$$\chi_X := (1 \xrightarrow{\eta} X \otimes X^\vee \xrightarrow{\simeq} X^\vee \otimes X \xrightarrow{\epsilon} 1)$$

where the symmetrizer in the center can be thought of as the analogue of our norm map in this context. For example, if $(\mathcal{C}, \otimes, 1) = (\mathsf{Sp}, \otimes, \mathbb{S})$, and $X \in \mathsf{Sp}^\diamond = \mathsf{Sp}^\omega$, then $\chi_X \in \pi_0\mathbb{S} = \mathbb{Z}$ is the **Euler characteristic** of the finite space X (here finite is in the sense of ω-compactness, which is equivalent to X being weakly equivalent to a finite CW complex).

On the other hand, in the context of $\mathcal{C}$-ambidexterity for a π-finite map $A \xrightarrow{q} B$ and a (co)complete ∞-category $\mathcal{C}$ (or at least finitely complete with sufficient limits and colimits so the following adjunctions exist), we look at the adjunctions $q_! \dashv q^* \dashv q_* : \mathcal{C}^B \to \mathcal{C}^A$ where q^* is pullback, $q_!$ is left Kan extension along q, and q_* is right Kan extension along q. When $B = \mathsf{pt}$ is the point, q^* becomes the diagonal, $q_! = \mathsf{colim}_A$, and $q_* = \mathsf{lim}_A$. The Norm map is then a natural comparison map (which need not always exist)

$$q_! \xRightarrow{\mathsf{Nm}_q} q_*$$

which in the case of $\mathcal{C}$-ambidexterity of q is an equivalence, along with all the associated norm maps for diagonal $A \to A \times_B A$ of q. The cardinality for a $\mathcal{C}$-ambidextrous map q then defines an analogue of the trace in the case of symmetric monoidal ∞-categories

$$\mathrm{id}_{\mathcal{C}} \xrightarrow{\eta} q_* q^* \xleftarrow[\simeq]{\mathsf{Nm}_q q^*} q_! q^* \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{C}}$$

For example, if $\mathcal{C} = \mathsf{Sp}$ is the infinity category of spectra, then we can use the natural equivalence

$$\mathsf{Fun}^L(\mathsf{Sp}, \mathsf{Sp}) \xrightarrow[\simeq]{- \circ \Sigma^\infty_+} \mathsf{Fun}^L(\mathcal{S}, \mathsf{Sp}) \xrightarrow[\simeq]{\mathsf{ev}_{\mathsf{pt}}} \mathsf{Sp}$$

(c.f. Universality of Multiplicative Infinite Loop Space Machines (Gepner, Groth, Nikolaus) > ^5eab9d) to observe that $\mathrm{id}_{\mathsf{Sp}}$ being cocontinuous means we can write it as $\mathbb{S} \otimes -$, so that

$$\pi_0 \mathsf{End}(\mathrm{id}_{\mathsf{Sp}}) \cong \pi_0 \mathsf{End}_{\mathsf{Sp}}(\mathbb{S}) = \mathbb{Z}$$

Thus, for any π-finite map, the Sp-cardinality of $q : A \to B$ corresponds to an integer, where for $X \in \mathsf{Sp}$, $|q|_X : X \to X$ is given by the composite

$$X \xrightarrow{\simeq} \mathbb{S} \otimes X \xrightarrow{(|q|_X \cdot) \otimes X} \mathbb{S} \otimes X \xrightarrow{\simeq} X$$

where we're identifying $|q|_X$ with the integer value.

Examples in Representation Theory

To begin let's consider the case of G a finite group so that $A = BG$ is a 1-finite space, and take $\mathcal{C} = R\mathsf{Mod}$ for a commutative unital ring R. Then $\mathcal{C}^A = R[G]\mathsf{Mod}$ is the category of R-valued G-representations for a commutative ring R. The map $q^* : R\mathsf{Mod} \to R[G]\mathsf{Mod}$ is given by sending an R-module to the trivial representation associated to it. On the other hand, $q_! M = \mathsf{colim}_{BG} M = M_G = M/(m \sim gm)$ sends a G-representation to the R-module of G-orbits, and $q_* M = \mathsf{lim}_{BG} M = M^G$ sends a G-representation to the R-module of G-fixed points. We then have a natural norm map

$$\mathsf{Nm}_G : M_G \to M^G, \ [m] \mapsto \sum_{g \in G} g \cdot m$$

The kernel of this map consists of those G-orbits such that $\sum_{g\in G} g\cdot m = 0$, while the image always at least contains $|G|M^G$. The norm map fits in the **Tate cohomology groups** which are defined by

$$\widehat{H}^i(G;M) := \begin{cases} H^i(G;M) & i \geq 1 \\ \mathsf{coker}(\mathsf{Nm}_G) & i = 0 \\ \mathsf{ker}(\mathsf{Nm}_G) & i = -1 \\ H_{-i-1}(G;M) & i \leq -2 \end{cases}$$

Recall here that $(-)^G = \mathsf{Hom}_{R[G]}(R,-)$, and that $H^n(G;-) := \mathbb{R}^n\mathsf{Hom}_{R[G]}(R,-) = \mathsf{Ext}^n_{R[G]}(R,-)$ are the right derived functors of the fixed point functor, while $(-)_G = R\otimes_{R[G]} -$, and $H_n(G;-) = \mathbb{L}^n(R\otimes_{R[G]} -) = \mathsf{Tor}_n^{R[G]}(R,-)$ are the left derived functors. We can also describe the group cohomology as the cohomology of the cochain complex associated to the simplicial R-module $\mathsf{Fun}(G^{(-)}, M)$, with face operators given by multiplying arguments internally, or acting on the left/right (with right action being trivial), and degeneracies given by inserting identities.

Example

Consider the case of $G = \mathbb{Z}/p$ and $R = \mathbb{Z}$. Then $\mathsf{Nm}_{\mathbb{Z}/p} : \mathbb{Z} \to \mathbb{Z}$ is just multiplication by p, implying that $\ker(\mathsf{Nm}_{\mathbb{Z}/p}) = 0$ but $\mathsf{coker}(\mathsf{Nm}_{\mathbb{Z}/p}) = \mathbb{Z}/p$.

Example

If R is a commutative ring and G is a group with $|G| \in R^\times$, then $\mathsf{Nm}_G : R \to R$ is multiplication by $|G|$, and hence is an isomorphism. In particular, the norm map for any constant representation is an isomorphism.

Example

If $G = \mathbb{Z}/4$, $R = \mathbb{C}$, and $M = \mathbb{C}$ with the action given by the inclusion $\mathbb{Z}/4 \xrightarrow{e^{i\pi t/2}} S^1 \subseteq \mathbb{C}$, then $M_G = \{0\} \cup \bigcup_{t\in[0,\pi/2)} e^{2\pi i t}\mathbb{R}_+$ is the space of homotopy orbits, while $M^G = \{0\}$ is the space of homotopy fixed points, so we would never have the norm map being an isomorphism.

Examples in Stable Homotopy Theory

Let G be a finite group and let $q : BG \to *$ be the unique 1-finite map of spaces. Let $\mathcal{C} = \mathsf{Sp}$ be the ∞-category of spectra so that $q_! = (-)_{hG}$ is the *homotopy orbits* functor and $q_* = (-)^{hG}$ is the homotopy fixed points functor. Equivalently, $q_* = \mathsf{Map}_{BG}(EG, -)$ and $q_! = EG \otimes_{BG} -$, where here we're using that Sp is tensored and cotensored over $\mathcal{S}$, being complete and cocomplete. Explicitly, for a spectrum X, $q_* X = F^G(EG_+, X)$ is the G-equivariant mapping spectrum and $q_! X = (\Sigma^\infty_+ EG \otimes X)/\Sigma^\infty_+ BG$ with diagonal action.

In this situation the Tate construction measures the defect for BG being Sp-ambidextrous:

$$X^{tG} = \mathsf{hocofib}(X_{hG} \xrightarrow{\mathsf{Nm}_G} X^{hG})$$

Here for M a $\mathbb{Z}[G]$-module, the Tate construction HM^{tG} has homotopy groups recovering the Tate cohomology

$$\pi_*(HM^{tG}) \cong \widehat{H}^{-*}(G; M)$$

References

1. Carmeli, S., Schlank, T.M., Yanovski, L.: Ambidexterity and Height, http://arxiv.org/abs/2007.13089 , (2020) ↩
2. Cnossen, Bastiaan, Tobias Lenz, and Sil Linskens. “Parametrized (Higher) Semiadditivity and the Universality of Spans.” arXiv:2403.07676. Preprint, arXiv, September 27, 2024. https://doi.org/10.48550/arXiv.2403.07676 . ↩
3. Carmeli, Shachar, Tomer M. Schlank, and Lior Yanovski. “Ambidexterity in Chromatic Homotopy Theory.” arXiv:1811.02057. Preprint, arXiv, September 16, 2020. https://doi.org/10.48550/arXiv.1811.02057 . ↩ ↩
4. Riehl, Emily, and Dominic Verity. *Elements of ∞-Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022. https://doi.org/10.1017/9781108936880 . ↩