The Langlands Program: Piecing Together a Bridge between Fields

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(they/them)

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SUM-C Summer Seminar Series

Roadmap

What is the Langlands Program about?

What is the LLC?

Geometric Perspective

- An interesting example

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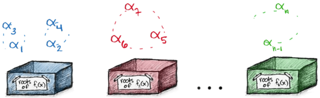
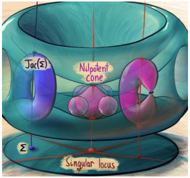
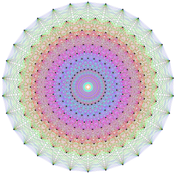


(a) Algebraic Number Theory



(b) Algebraic Geometry



(c) Representation Theory

Figure: The Langlands Program: bridging fields

What is the Local Langlands Correspondence?

The LLC

Fix G a connected reductive algebraic group over a p-adic field, $F/\mathbb{Q}_p$

$$\left\{ \begin{array}{c} \text{Admissible irreducible} \\ \text{representations of } G \end{array} \right\} \Big/ iso. \xrightarrow{\mathbf{r}} \left\{ \begin{array}{c} \text{Langlands parameters} \\ \phi : W_F \times \mathsf{SL}_2(\mathbb{C}) \to {}^L G \end{array} \right\} \Big/ conj.$$

p-Adic Fields

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Fix $p \in \mathbb{Z}^+$ a prime. We denote by $\mathbb{Q}_p$ the analytic completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$ defined by $\left|\frac{a}{b}p^r\right|_p = p^{-r}$ for $a, b, r \in \mathbb{Z}$, with $p \nmid a, b$.

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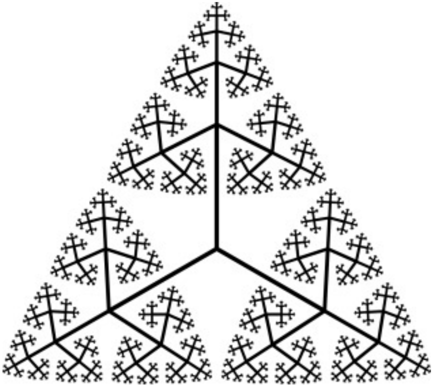

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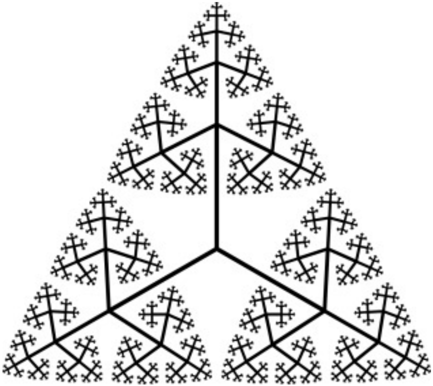
$x \in \mathbb{Q}_p$ can be written

$$x = \sum_{i=k}^{\infty} a_i p^i$$

for $k \in \mathbb{Z}$ and $a_i \in \{0, 1, ..., p-1\}$.



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How do you represent a group?

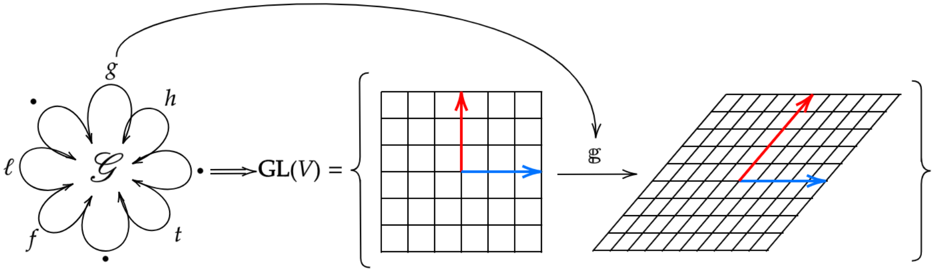

Figure: Representations of groups

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Galois Group

Let E be a field obtained by appending roots to F. Then let $\mathrm{Gal}(E/F)$ denote the group of field automorphisms of E fixing points in F.

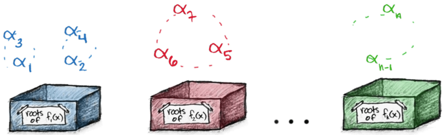

Figure: Permuting roots

Separable Closure (Informally)

The separable closure, $\overline{F}$, of F consists of all roots of "multiplicity free" polynomials in F.

Playing with roots

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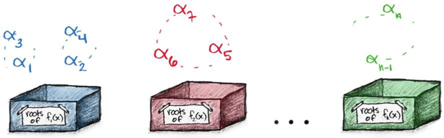

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Integer Rings and Residue Fields

Integer Rings and Residues

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Example:

- $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p = \left\{\sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, ..., p-1\}\right\}$, and $k_{\mathbb{Q}_p} = \mathbb{F}_p$

The Weil Group

$$1 \longrightarrow I_F \longrightarrow \mathsf{Gal}(\overline{F}/F) \longrightarrow \mathsf{Gal}(\overline{\mathbb{F}_{q_F}}/\mathbb{F}_{q_F}) \longrightarrow 1$$

The Weil Group

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] \arrow[d, equal] & \mathsf{Gal}(\overline{F}/F) \arrow[r] & \mathsf{Gal}(\overline{\mathbb{F}_{q_F}}/\mathbb{F}_{q_F}) \arrow[r] & 1 \\
1 \arrow[r] & I_F \arrow[r] & W_F \arrow[r] \arrow[u, hook] & W_{k_F} \arrow[r] \arrow[u, hook] \arrow[l, bend right] \arrow[d, equal] & 1 \\
 & & & \langle x \mapsto x^{q_F} \rangle &
\end{tikzcd}$$

- $W_F \curvearrowright \widehat{G}$ gives

$$
\begin{tikzcd}
1 \arrow[r] & \mathsf{SL}_2(\mathbb{C}) \arrow[r] \arrow[d] & W_F \times \mathsf{SL}_2(\mathbb{C}) \arrow[r] \arrow[d, "\phi"] \arrow[dl, "\phi^\circ"'] & W_F \arrow[r] \arrow[d, equal] \arrow[l, bend right] & 1 \\
1 \arrow[r] & \widehat{G} \arrow[r] & {}^L G \arrow[r] \arrow[d, equal] & W_F \arrow[r] \arrow[l, bend right] & 1 \\
 & & \widehat{G} \rtimes W_F & &
\end{tikzcd}
$$

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$$\Big\Downarrow$$

$$\prod_{\lambda} (G/F) \leftrightarrow \mathsf{Per}_{H_\lambda}(V_\lambda)^{simple} \Big/ iso.$$

Infinitesimal Parameters

- Let $\phi : W_F \times \mathbf{SL}_2(\mathbb{C}) \to {}^L G$ be a Langlands parameter

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Infinitesimal Parameter

The infinitesimal parameter associated with ϕ is defined by

$$\begin{aligned}\lambda_\phi :& W_F \to {}^LG \\ & w \mapsto \phi\left(w, \begin{pmatrix} |w|_F^{1/2} & 0 \\ 0 & |w|_F^{-1/2} \end{pmatrix}\right)\end{aligned}$$

Moduli Space

- Fix an infinitesimal parameter $\lambda : W_F \to {}^L G$ moving forward

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Vogan Variety

Define the centralizer

$$Z_{\widehat{G}}(\lambda(I_F)) := \{g \in \widehat{G} : (g \rtimes 1)\lambda(w)(g \rtimes 1)^{-1} = \lambda(w), \forall w \in I_F\}$$

Then the Vogan Variety associated with λ is

$$V_\lambda := \{x \in \mathrm{Lie}\ Z_{\widehat{G}}(\lambda(I_F)) : \lambda(w)x\lambda(w)^{-1} = |w|_F x, \forall w \in W_F\}$$

along with an action by $H_\lambda := Z_{\widehat{G}}(\lambda(W_F))$.

Sheaves

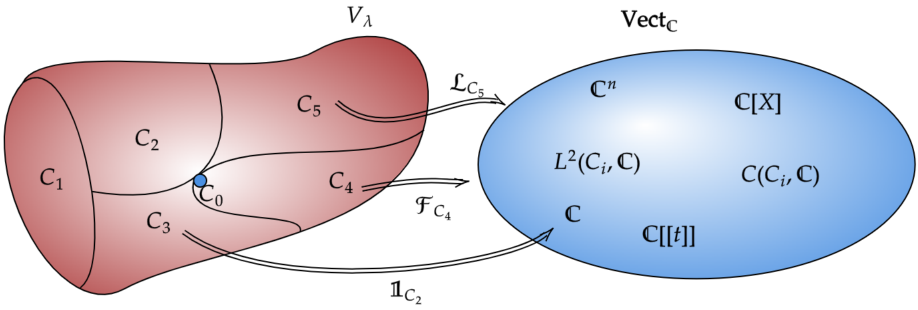

Figure: Sheaves on a Vogan

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$$\Downarrow$$

$$\prod_{\lambda} (G/F) \leftrightarrow \mathrm{Per}_{H_\lambda}(V_\lambda)^{simple} \big/ iso.$$

We have a canonical bijection

$$\mathsf{Per}_{H_\lambda}(V_\lambda)^{simple}/iso. \leftrightarrow \{(C,\rho) : C \subseteq V_\lambda\ H_\lambda \text{ orb},\ \rho \in \mathsf{Irrep}(\pi_1(C,x_0)_{H_\lambda})\}$$

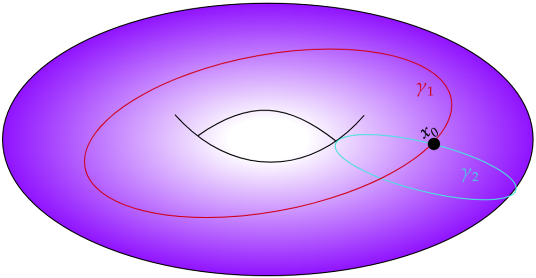

Figure: Fundamental group of a torus

$\mathbf{GL}_{1+k+1}(F)$ Case

- The image of frobenius for λ is

$$\lambda(Fr) = \begin{pmatrix} q_F^1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & q_F^{-1} \end{pmatrix} \rtimes Fr$$

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- The Vogan is

$$V_\lambda = \left\{ \begin{pmatrix} 0 & x_1 & \cdots & x_k & 0 \\ 0 & 0 & \cdots & 0 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & y_k \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} : x_i, y_i \in \mathbb{C} \right\} \cong M_{1,k}(\mathbb{C}) \times M_{k,1}(\mathbb{C})$$

$$\cong \mathrm{Hom}(E_1, E_{q_F^1}) \times \mathrm{Hom}(E_{q_F^{-1}}, E_1)$$

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$\mathbf{GL}_{1+k+1}(F)$ Case

- Our group is $H_\lambda \cong \mathbf{GL}_1(\mathbb{C}) \times \mathbf{GL}_k(\mathbb{C}) \times \mathbf{GL}_1(\mathbb{C})$
- We have five orbits:

$$\begin{tikzcd}
& C_t \arrow[d, dash] & \\
& C_m \arrow[dl, dash] \arrow[dr, dash] & \\
C_l \arrow[dr, dash] & & C_r \arrow[dl, dash] \\
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\end{tikzcd}$$

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$\mathbf{GL}_{1+k+1}(F)$ Case

- $\pi_1(C, x)_{H_\lambda} \cong \{0\}$ for all orbits, so

  $\mathsf{Per}_{H_\lambda}(V_\lambda)^{simple}/iso. \cong \{IC(C_0, \mathbb{1}_{C_0}), IC(C_l, \mathbb{1}_{C_l}), IC(C_r, \mathbb{1}_{C_r}), IC(C_m, \mathbb{1}_{C_m}), IC(C_t, \mathbb{1}_{C_t})\}$

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m^λ_{geo}	$\vert_{C_0}$	$\vert_{C_l}$	$\vert_{C_r}$	$\vert_{C_m}$	$\vert_{C_t}$
$IC(C_0,\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0	0	0
$IC(C_l,\mathbb{1}_{C_l})$	$\mathbb{1}_{C_0}[k]$	$\mathbb{1}_{C_l}[k]$	0	0	0
$IC(C_r,\mathbb{1}_{C_r})$	$\mathbb{1}_{C_0}[k]$	0	$\mathbb{1}_{C_r}[k]$	0	0
$IC(C_m,\mathbb{1}_{C_m})$	?	?	?	$\mathbb{1}_{C_m}[2k-1]$	0
$IC(C_t,\mathbb{1}_{C_t})$	$\mathbb{1}_{C_0}[2k]$	$\mathbb{1}_{C_l}[2k]$	$\mathbb{1}_{C_r}[2k]$	$\mathbb{1}_{C_m}[2k]$	$\mathbb{1}_{C_t}[2k]$

Fixing Singularities: Resolutions

- We wish to find a smooth space $\widetilde{C_m}$ with a natural "nice" projection

$$\pi : \widetilde{C_m} \to C_m$$

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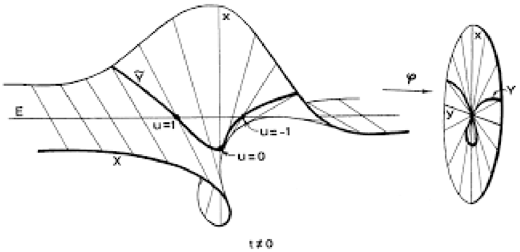
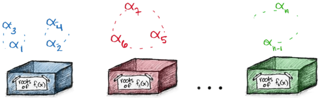


Figure: Resolution of Singularities through blow-up

To be continued ...

Thank you for your attention!



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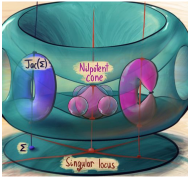
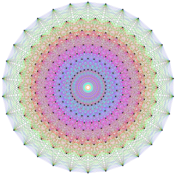


(b) Algebraic Geometry



(c) Representation Theory

Figure: The Langlands Program: bridging fields

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[6] D. A. Vogan. "The local Langlands conjecture". In: 1993.