Building Fourier Analysis for Groups Through Duality

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Abstract

A fundamental tool in modern applied and pure mathematics is the Fourier transform for real functions and functions on special classes of groups more generally. Since its initial formulation in the early 1800s, the Fourier transform has been vastly generalized and has found important uses in imaging, signal analysis, and even representation theory. In this paper we will develop some of the basic theory of Fourier analysis for abelian locally compact groups. To achieve this we will begin by briefly introducing and developing some theory for locally compact groups and their Haar measures. We will also define the convolution product on L^1 space and describe its relation to the dual group of a locally compact group. Finally we will prove and sketch a number of important lemmas which will imply Pontryagin duality for abelian locally compact groups, and show how Pontryagin duality allows for a simple proof of the inversion formula for the Fourier transform.

1 Introduction

Since its introduction by Jean Baptiste Joseph Fourier in the early 1800s, the Fourier transform for real-valued functions has proved instrumental to a number of developments in both applied and pure mathematics [Dom16]. In applied settings the Fourier transform has been used in a wide variety of fields ranging from signal processing to quantum mechanics, and even computer tomography imaging [Dom16]. On the other hand, the Fourier transform has also found important uses in pure mathematical fields such as representation theory and number theory through extensions of the Fourier transform to groups [DE14].

Although the theory of abstract integration on sets had been developed in the late 1800s to early 1900s, when integrating over objects with additional structure such as groups it was highly desirable to have integrals which respected that structure. The Lebesgue measure and theory of integration for Euclidean space is a perfect example of a theory of integration for an abelian group which respects the group structure in the sense that the measure is invariant under translation. Similar such invariant measures were developed for well-structured groups, such as the development of invariant integrals for Lie groups by Adolf Hurwitz in 1897 [HR94], but it wasn't until Alfréd Haar in 1933 when the existence of invariant measures was extended to a far larger class of groups [HR94].

In this paper we will develop the theory of invariant integration on locally compact groups, with the goal of introducing the Fourier transform for such groups as well as Pontryagin duality for abelian locally compact groups. We will begin by providing a brief introduction to the theory of topological groups before defining Haar measures and proving their uniqueness. In order to develop the theory of Pontryagin duality we will provide a brief investigation into the Banach algebra structure on the space of L^1 functions over an abelian locally compact group A, including a treatment of its interaction with the dual group of A. With these preliminaries developed we will prove and sketch important results such as the Fourier inversion formula for integrable functions that are represented by complex Radon measures. Finally, we will state Pontryagin duality and use it to show that the Fourier transform can be inverted for L^1 functions with L^1 Fourier transform. Throughout, we will provide examples of special groups and their duals that appear in the modern theory of Fourier transforms in order to help illuminate and motivate the concepts being discussed.

2 Background Information

In this paper we assume a basic familiarity with the theory of groups and topological spaces as preliminaries. For those not familiar with these subjects we recommend the text by Dummit & Foote for an introductory abstract algebra reference [DF04] and the text by Munkres for an introductory topology reference [Mun18]. With these references in mind we can begin developing the pre-requisite theory of topological groups needed for a study of Pontryagin duality.

2.1 Topological Groups

Instead of integrating over arbitrary groups as measurable spaces, in order to assist with the development of our integration and fourier analysis theory we restrict to groups with additional structure. In particular, we want to consider groups which are topological spaces in their own right so that our measures will be Borel measures on these spaces.

Definition 2.1 [Mun18, p. 143] A group G is said to be a topological group if G also is equipped with a topology τ such that the multiplication and inversion maps, $m : G \times G \to G$ and $i : G \to G$, are continuous.

The simple continuity requirements in this definition of topological group ensures that group elements act by homeomorphisms on the group G.

Proposition 2.2 Let G be a topological group. Then for any $g \in G$, the map $l_g : G \to G$ given by $l_g(g') := gg'$ for all $g' \in G$ is a homeomorphism.

Proof. Fix an element $g \in G$. Then we have a map $\iota_g : G \to G \times G$ given by $\iota_g(g') = (g, g')$ for all $g' \in G$. This map is continuous since the open sets $A \times B \subseteq G \times G$ form a basis for the product topology on $G \times G$, and $\iota_g^{-1}(A \times B)$ is either B if $g \in A$ or is $\emptyset$ if $g \notin A$, both of which are open. It follows that $l_g : G \to G$ as described in the proposition statement is continuous since it can be written as the composite $m \circ \iota_g$ of the continuous maps m and ι_g.
Since $g \in G$ was arbitrary, we also have that $l_{g^{-1}}$ is a continuous map. But for any $g' \in G$,

$$l_{g^{-1}}(l_g(g')) = l_{g^{-1}}(gg') = g^{-1}gg' = g'$$

and

$$l_g(l_{g^{-1}}(g')) = l_g(g^{-1}g') = gg^{-1}g' = g'$$

so l_g and $l_{g^{-1}}$ are inverse. It follows that l_g is a homeomorphism, as claimed. ■

Note that an identical argument to that given in the proof of Proposition 2.2 implies that for $g \in G$, the map $r_g : G \to G$ given by $r_g(g') = g'g$ is a homeomorphism. Additionally, since $i^2(g) = i(g^{-1}) = g$ for $g \in G$, the inversion operation i on G is also a homeomorphism, being its own inverse.

Proposition 2.2 implies that a topological group must have a very structured topology as topological features can be translated through the group via multiplication by group elements. Indeed, if $H \subseteq G$ is any subspace, then for any $g \in G$ we have that the translate gH is homeomorphic to H, and therefore has identical topological properties. For example, if H is a non-empty open subset of G then gH is open in G for all $g \in G$, which implies that the translates $\{gH\}_{g\in G}$ form an open cover of G. Indeed this collection covers G since H being non-empty implies we have some $g' \in H$, so that for any $g \in G$, $g = gg'^{-1}g' \in (gg'^{-1})H$.

Another important consequence of Proposition 2.2 is that all open subgroups of a topological group G are closed. Indeed, if H is a subgroup of G, since the cosets gH partition G we can write $G\backslash H = \bigcup_{g\in G\backslash H} gH$. But since H is open, so is each gH for $g \in G\backslash H$, so $G\backslash H$ itself is open, being the union of open sets. Hence, by definition H is closed in G.

Although topological groups are already incredibly structured, for our purposes we will assume that all of our topological groups satisfy two additional properties. First, we assume that all of our topological groups are Hausdorff, so we can separate points. Secondly, we also require that all of our topological groups are locally compact. Since our topological groups are all Hausdorff, being locally compact is equivalent to the following [BBT20, Thm. 2.19] definition.

Definition 2.3 A Hausdorff topological space X is said to locally compact if for each $x \in X$ and each open neighborhood U of x, there exists another open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

A number of topological groups of interest satisfy these requirements. For our current work in Fourier analysis four such groups which are notable are the groups $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, with group structure given by addition, as well as $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ with group structure given by complex multiplication.

Next, in order to assist with our development of an integration theory on locally compact groups we require certain topological properties that they satisfy. The first result will entail the existence of open subgroups of a locally compact group G that will be σ-finite with respect to the measures on G in Section 2.2. This existence result will be important for the application of theorems on double integration which rely on the use of σ-finite measure spaces. For the sake of space we do not proof this result hear, and instead refer the reader to [DE14, Prop. 1.2.1].

Proposition 2.4 Let G be a locally compact group. Then G has a σ-compact open subgroup and the union of countably many such σ-compact open subgroups itself generates a σ-compact open subgroup.

The second topological property for locally compact groups which will be useful for the proof of Pontryagin duality is a relation between locally compact and closed subgroups of a locally compact group. Once again in order to save space for the proof of Pontryagin duality we refer the reader to [Fol94, Lem. 4.31] for a proof of the following lemma.

Lemma 2.5 Let G be a locally compact group. Then any locally compact subgroup H of G must be closed in G.

With these topological pre-liminaries and conventions set, we can now move to investigating the Haar measure on locally compact Hausdorff topological groups. For Pontryagin duality we only consider abelian locally compact groups, so we restrict to this setting moving forward. Throughout we fix the notation A for an abelian locally compact group.

2.2 Haar Measures

As mentioned previously, a classical example of a locally compact topological group is the real line, $\mathbb{R}$, equipped with addition. In addition to this topological group structure, $\mathbb{R}$ is also a measure space when equipped with the Lebesgue measure m. One of the most important properties of the Lebesgue measure, and the one that is most relevant for our current work, is the fact that the Lebesgue measure is translation invariant from its construction [Tay06, Chap. 2]. Explicitly this means that for any measurable subset $E \subseteq \mathbb{R}$ and any real number $x \in \mathbb{R}$,

$$m(x+E) = m(E)$$

where $x + E = \{x + e \in \mathbb{R} : e \in E\}$. The Lebesgue measure is a special case of a large class of measures on locally compact groups, called Haar measures. Haar measures themselves are a special case of Radon measures.

Definition 2.6 [Fol94, p. xii] Let X be a locally compact Hausdorff space. A Radon measure on X is a Borel measure on X which is outer regular, inner regular for open sets, and has finite measure for compact sets.

A left (respectively right) Haar measure, μ, on a locally compact group G is then simply a non-zero Radon measure such that for any $g \in G$ and any Borel measurable subset $M \subseteq G$, $\mu(gM) = \mu(M)$ (respectively $\mu(Mg) = \mu(M)$) where $gM = \{gm \in G \,|\, m \in M\}$ [Coh13, p. 285]. Note that from Proposition 2.2, since continuous functions are Borel measurable, for each Borel measurable subset $M \subseteq G$, gM (respectively Mg) is still Borel measurable in G for any $g \in G$, so the definition is well-posed. Since we are considering abelian locally compact groups A, left and right Haar measures are equivalent, so we just refer to Haar measures moving forward.

To provide a concrete example of a Haar measure on a locally compact group we can return to the example of the group $\mathbb{Z}^n$ under addition.

Example 2.1 (Haar Measure on $\mathbb{Z}^n$) :

The group $\mathbb{Z}^n$ has a natural Haar measure given by the counting measure, $\mu_{\mathbb{N}}$. Explicitly, for a subset $I \subseteq \mathbb{Z}^n$, $\mu_{\mathbb{N}}(I) = \#(I)$, where $\#(I)$ denotes the cardinality of I.

Note that $\mathbb{Z}^n$ being a discrete subgroup of $\mathbb{R}^n$ implies that all subsets are Borel measurable, and so $\mu_{\mathbb{N}}$ is defined on all subsets of $\mathbb{Z}^n$. Further, $\mu_{\mathbb{N}}$ is translation invariant since for any $\vec{m} \in \mathbb{Z}^n$, by Proposition 2.2 addition by $\vec{m}$ is a homeomorphism. In particular, this implies that for any subset $I \subseteq \mathbb{Z}^n$, I and $I + \vec{m}$ are in bijection, so

$$\mu_{\mathbb{N}}(I) = \#(I) = \#(I+\vec{m}) = \mu_{\mathbb{N}}(I+\vec{m})$$

It is a well-known result, first proved by Haar for locally compact groups with countable bases for their topologies, and later generalized to all locally compact groups by Weil [HR94], that Haar measures always exist on locally compact topological groups. Modern proofs of this existence follow a similar style to the construction of the Lebesgue measure on $\mathbb{R}$, where first an invariant outer measure is constructed and then it is proved to satisfy of desirable properties corresponding to it being a Radon measure [Coh13, Sec. 9.2]. As a rigorous proof of this result involves lemmas that go beyond the scope of this paper, we will assume this result and instead demonstrate certain valuable properties of Haar measures to help motivate their study.

First, note that if $f : A \to \mathbb{C}$ is a Borel measurable function on A, then $f \circ l_a$ is Borel measurable for any $a \in A$ since the composition of measurable functions is measurable. We can consider

$f \circ l_a$ to be a kind of translation of f by $a \in A$. An important property of Haar measures is that integration is invariant with respect to translations of measurable functions in this sense.

Proposition 2.7 [Coh13, p. 286] Let $f : A \to \mathbb{C}$ be a Borel measurable function on A and let μ be a Haar measure on A. Then for any $a \in A$,

$$\int_A f \circ l_a d\mu = \int_A f d\mu$$

Proof. To begin, let $M \subseteq A$ be Borel measurable and let $a \in A$. Then for $a' \in A$, $\chi_M \circ l_a(a') = \chi_M(aa')$ is 1 if $aa' \in M$, and zero otherwise. But $aa' \in M$ if and only if $a' \in a^{-1}M$, so $\chi_M \circ l_a = \chi_{a^{-1}M}$. Since μ is a Haar measure it follows that

$$\int_A \chi_M \circ l_a d\mu = \int_A \chi_{a^{-1}M} d\mu = \mu(a^{-1}M) = \mu(M) = \int_A \chi_M d\mu$$

as desired. The case for non-negative simple functions follows by linearity of integration. Then by the Monotone Convergence Theorem [Tay06, Thm. 3.4] we have that the result holds for any non-negative measurable function by taking a limit of simple functions. Since we can write any measurable function f as a sum $f = f^+_{Re} - f^-_{Re} + if^+_{Im} - if^-_{Im}$ of non-negative measurable functions, it follows that the result holds for all measurable functions that integration is defined on. ■

Note that since A is abelian, for any $a, a' \in A$, $l_a(a') = aa' = a'a = r_a(a')$, so left and right translations are equivalent in A. In other words, the result of Proposition 2.7 holds equally well using right translations.

Another incredibly valuable property of Haar measures is that they are unique up to scalar multiple. To prove this partial uniqueness we first require the following lemma on integration with respect to continuous non-negative functions.

Lemma 2.8 [Coh13, Lem 9.2.5] If μ is a Haar measure on A, then each non-empty open set has positive measure and each continuous $f : A \to [0, \infty)$ that isn't identically zero satisfies

$$\int_A f d\mu > 0$$

Proof. Since by definition μ is a non-zero Radon measure, the measure of any open set can be approximated from below by the measures of compact sets, and the measure of any measurable set can be approximated from above by open sets. In particular, as $\mu(A) > 0$, since μ is non-zero, there must exist a compact set K such that $\mu(K) > 0$. Let U be any non-empty open set in A. Then from the discussion after Proposition 2.2, $\{aU\}_{a \in A}$ is an open cover of K. But K is compact, so there exists $a_1, ..., a_n \in A$ such that $a_1U, ..., a_nU$ covers K. Then since μ is a Haar measure

$$0 < \mu(K) \leq \sum_{i=1}^{n} \mu(a_1U) = n\mu(U)$$

so $\mu(U) \geq \frac{\mu(K)}{n} > 0$.

To prove the second claim let $f : A \to [0, \infty)$ be a continuous map that is not identically zero. Then there exists $\epsilon > 0$ such that $U := f^{-1}((\epsilon, \infty))$ is a non-empty open set. In particular, this implies that $0 \leq \epsilon\chi_U \leq f$. Since U is a non-empty open set the first part of the proof implies that

$$0 < \epsilon\mu(U) = \int_A \epsilon\chi_U d\mu \leq \int_A f d\mu$$

using the monotonicity of integration [Coh13, Prop. 2.3.1]. ■

With Lemma 2.8 we can now show that Haar measures on a locally compact group are unique up to scalar multiple. In order to not rely on too many results that are beyond the scope of this paper we will only prove the uniqueness for locally compact groups with countable bases. The general proof is described in [Coh13, Thm. 9.2.6].

Proposition 2.9 Suppose A has a countable base for its topology. Let μ and ν be Haar measures on A. Then there exists a real number $c > 0$ such that $\mu = c\nu$.

Proof. First, I claim that A with any Haar measure μ is a σ-finite measure space. Observe that since A is locally compact for each $a \in A$ we have a compact set K_a and an open set U_a such that $a \in U_a \subseteq K_a$. Since μ is a Haar measure, in particular it is a Radon measure, so $\mu(U_a) \leq \mu(K_a) < \infty$. Then $A = \bigcup_{a\in A} U_a$ is an open cover of A. But since A has a countable base, there exists a sequence $a_n \in A$, $n \geq 1$, such that $A = \bigcup_{n\geq 1} U_{a_n}$. Additionally, $\mu(U_{a_n}) < \infty$ for each $n \geq 1$, so A is σ-finite, as claimed.

Now, let μ and ν be two Haar measures on A. Since μ and ν are Radon measures they are lower regular for open sets and upper regular for all measurable sets, so it is sufficient to prove the claim for compact sets. Let $K \subseteq A$ be a compact set. As Haar measures are Radon measures, $\mu(A), \nu(A) < \infty$. Now, since A is locally compact there exists a non-empty open set $U \subseteq A$ with compact closure, $\overline{U}$. By monotonicity this implies that $\mu(U) \leq \mu(\overline{U}) < \infty$ and $\nu(U) \leq \nu(\overline{U}) < \infty$. By Lemma 2.8 we also that have $\mu(U), \nu(U) > 0$.

Note that we have the non-negative measurable function $\chi_{U\times K} : A \times A \to A$ given by $\chi_{U\times K}(a, a') = \chi_U(a)\chi_K(a')$. Since A is a σ-finite measure space with respect to both μ and ν, we can apply Tonelli's Theorem [Tay06, Thm. 6.3]. It follows that we can perform the following computation

$$\begin{aligned}
\mu(K)\nu(U) &= \int_A \int_A \chi_K(a) d\mu(a) \chi_U(a') d\nu(a') \\
&= \int_A \int_A \chi_K(a'a) d\mu(a) \chi_U(a') d\nu(a') \\
&\qquad \text{(by Proposition 2.7 applied to the inner integral)} \\
&= \int_A \int_A \chi_K(a'a) \chi_U(a') d\nu(a') d\mu(a) \qquad \text{(by Tonelli's Theorem)} \\
&= \int_A \int_A \chi_K(aa') \chi_U(a'a^{-1}a) d\nu(a') d\mu(a) \\
&= \int_A \int_A \chi_{U\times K} \circ l_{(a,a)}(a'a^{-1}, a') d\nu(a') d\mu(a) \qquad \text{(using the fact that } A \text{ is abelian)} \\
&= \int_A \int_A \chi_{U\times K}(a'a^{-1}, a') d\nu(a') d\mu(a) \\
&\qquad \text{(by Proposition 2.7 applied to the product group } A \times A) \\
&= \int_A \int_A \chi_K(a') \chi_U \circ l_{a'}(a^{-1}) d\nu(a') d\mu(a) \\
&= \int_A \int_A \chi_U \circ l_{a'}(a^{-1}) d\mu(a) \chi_K(a') d\nu(a') \qquad \text{(by Tonelli's Theorem)}
\end{aligned}$$

$$= \int_A \int_A \chi_U(a^{-1}) d\mu(a) \chi_K(a') d\nu(a')$$

(by Proposition 2.7 applied to the inner integral)

$$= \nu(K) \int_A \chi_U(a^{-1}) d\mu(a)$$

Note that $\chi_U(a^{-1}) = 1$ if and only if $a \in U^{-1} = \{b^{-1} \in A : b \in U\}$, so $\chi_U(a^{-1}) = \chi_{U^{-1}}(a)$. It follows that $\int_A \chi_U(a^{-1}) d\mu(a) = \mu(U^{-1}) > 0$ by Lemma 2.8 since U^{-1} is a non-empty open set as the inversion map for a topological group is a homeomorphism. Therefore, we have that for $c = \frac{\mu(U^{-1})}{\nu(U)} > 0$

$$\mu(K) = c\nu(K)$$

As K was an arbitrary compact set in A and μ and ν are Radon measures, it follows that $\mu = c\nu$, as desired. ■

In addition to the uniqueness of the Haar measure, Lemma 2.8 also allows us to bound the support of integrable functions on locally compact groups equipped with Haar measures. This bound will be essential for the definition of the convolution product in the next section.

Lemma 2.10 [DE14, Cor. 1.3.6] Let G be a locally compact group with Haar measure μ. Then if $f \in L^1(G)$, there exists a σ-compact open set containing the support of f.

Proof. Let $f \in L^1(G)$. Let $C = \{g \in G \mid f(g) \neq 0\}$, so $\overline{C} = \operatorname{supp}(f)$. We begin by showing that C is contained in a σ-compact open set. For each $n \in \mathbb{N}$ let $C_n := \{g \in G \mid f(g) > 1/n\}$, so $C = \bigcup_{n \geq 1} C_n$. Then it is sufficient to show that each C_n is covered by a σ-compact open set, since a countable union of countable unions is again a countable union.

Fix $n \in \mathbb{N}$. Since $f \in L^1(G)$ we must have that $\mu(C_n) < \infty$. Since μ is a Radon measure, and hence outer regular, there exists an open set U containing C_n such that $\mu(U) < \mu(C_n) + 1 < \infty$. Recall that G has a σ-compact open subgroup H by Proposition 2.4. Since H is a subgroup of G its cosets partition G, so we have a subset $D \subseteq G$ such that

$$G = \coprod_{g \in D} gH$$

Note that this implies $U = \coprod_{g \in D} (gH \cap U)$. I claim that U only intersects at most countably many elements of D. Indeed, for each $n \in \mathbb{N}$ we can consider the collection $\{g \in D : \mu(gH \cap U) > 1/n\}$. Note that all $g \in D$ such that $gH \cap U \neq \emptyset$ appear in one of these sets by Lemma 2.8. If the set of $g \in D$ such that $gH \cap U \neq \emptyset$ is uncountable then for some $n \in \mathbb{N}$ we must have that $\{g \in D : \mu(gH \cap U) > 1/n\}$ is infinite, as otherwise the union of a countable number of finite sets would be countable. In particular, this implies that we have $g_1, g_2, \ldots \in \{g \in D : \mu(gH \cap U) > 1/n\}$, so by countable additivity and monotonicity of measures,

$$\mu(U) \geq \mu\left(\coprod_{m \geq 1} (g_m H \cap U)\right) = \sum_{m \geq 1} \mu(g_m H \cap U) \geq \sum_{m \geq 1} \frac{1}{n} = \infty$$

contradicting the fact that $\mu(U) < \infty$.

Therefore, we have a countable set $g_1, g_2, \ldots \in D$ such that $U = \coprod_{n \geq 1} (g_n H \cap U)$. Since the H is σ-compact and the elements of G act by homeomorphisms by Proposition 2.2, $g_n H$ is σ-compact for every $n \geq 1$. Then by Proposition 2.4 the open subgroup generated by the union $\coprod_{n \geq 1} g_n H$ is σ-compact and it contains U. As U contains C_n it follows that for each $n \in \mathbb{N}$, C_n is covered by a σ-compact open subgroup, and so by our previous argument so is C.

Since open subgroups of a topological group are also closed, it follows that $\text{supp}(f) = \overline{C}$ is contained in a σ-compact open subgroup, as desired. ■

The uniqueness of the Haar measure along with the proof technique in Lemma 2.10 allows us to show that in our case where A is abelian, the Haar measure on A is not only translationally invariant, but is also invariant under inversion. Here we will write U^{-1} for the image $i(U) = \{u^{-1} \in A \mid u \in U\}$ of a subset $U \subseteq A$.

Corollary 2.11 For $f : A \to \mathbb{C}$ a Borel measurable function on A and let μ be a Haar measure on A. Then for any $a \in A$,

$$\int_A f \circ i d\mu = \int_A f d\mu$$

where $i : A \to A$ is the inversion of the group structure.

Proof. As in the proof of Proposition 2.7 it is sufficient to prove the claim for the case of characteristic functions, as the general case follows by linearity of integration, the Monotone Convergence Theorem, and approximation of measurable functions in terms of simple functions.

First, note that the function $\mu_i : \mathcal{B}(A) \to [0, \infty]$ defined on a Borel measurable set $M \subseteq A$ by $\mu_i(M) := \mu(M^{-1})$ is a measure since μ is a measure, and for $M_1, M_2, ... \subseteq A$ disjoint, $i\left(\coprod_{n\geq 1} M_n\right) = \coprod_{n\geq 1} i(M_n)$ since i is a homeomorphism. Further, we have that for $a \in A$ and $M \subseteq A$ measurable,

$$\mu_i(aM) = \mu((aM)^{-1}) = \mu(M^{-1}a^{-1}) = \mu(a^{-1}M^{-1}) = \mu(M^{-1}) = \mu_i(M)$$

since A is abelian and μ is a Haar measure. Thus μ_i is a Haar measure on A, so by the uniqueness of Haar measures [Coh13, Thm. 9.2.6] there exists $c > 0$ such that $\mu = c\mu_i$.

To determine c let $U \subseteq A$ be an open neighborhood of the identity element, e, in A with compact closure. Such a set exists since A is locally compact, and by Lemma 2.8 and the fact that μ is a Radon measure, $0 < \mu(U) \leq \mu(\overline{U}) < \infty$. Now, note that $U \cap U^{-1}$ is also an open neighborhood of e, and $i(U \cap U^{-1}) = U \cap U^{-1}$, so

$$\mu_i(U \cap U^{-1}) = \mu(U \cap U^{-1}) > 0$$

which implies that $c = 1$. Therefore $\int_A \chi_M d\mu = \mu(M) = \mu(M^{-1}) = \int_A \chi_M \circ i d\mu$ for any measurable set $M \subseteq A$, completing the proof. ■

Moving forward we will fix a Haar measure μ on the locally compact abelian group A.

3 Fourier Transform

Now that we have developed the basic theory of the integration on locally compact groups, we can begin defining Fourier analysis on such groups. To start we will construct the convolution product on $L^1(A)$ and define the dual group of A which naturally acts on $L^1(A)$ via the Fourier transform. Following this initial investigation we will prove and sketch important results related to the theory of the Fourier transform on locally compact groups. Finally we will show how Pontryagin duality for abelian locally compact groups gives an elegant proof of the Fourier inversion formula.

3.1 Convolution and the Dual Group

A central tool in Fourier analysis on groups comes from a generalization of the convolution for $L^1(\mathbb{R}^n)$ to a convolution on $L^1(A)$, where $L^1(A)$ consists of complex-valued integrable functions in A. The following result allows us to define a convolution product on $L^1(A)$ in a sensible way.

Proposition 3.1 [DE14, Thm. 1.6.2] Let $f, g \in L^1(A)$. Then the convolution

$$f * g(a) := \int_A f(b)g(b^{-1}a)d\mu(b)$$

exists μ-almost everywhere and setting $f * g$ to be 0 where the integral does not exist gives a function in $L^1(A)$ such that $||f * g||_{L^1} \leq ||f||_{L^1}||g||_{L^1}$.

Proof. Let $f, g \in L^1(A)$. Note that since A is a topological group the map $\varphi : A \times A \to A \times A$ given by $\varphi(a, b) = (a, a^{-1}b)$ for $a, b \in A$ is continuous as each of its components are continuous. Additionally, note that the map $f \times g : A \times A \to \mathbb{C} \times \mathbb{C}$ given by $(f \times g)(a, b) = (f(a), g(b))$ is measurable with respect to the product measures as it preserves measurable rectangles as each of f and g are measurable. Finally, as the multiplication $m : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ in the complex numbers is continuous, we have that composite map $\alpha := m \circ (f \times g) \circ \varphi : A \times A \to \mathbb{C}$ is Borel-measurable.

If $b \in A$ is fixed we have a map $i_b : A \to A \times A$ given by $i_b(a) = (a, b)$ for all $a \in A$. Note that i_b is continuous since for any basic open set $U \times V \subseteq A \times A$, $i_a^{-1}(U \times V)$ is either U if $b \in V$ or is $\emptyset$ if $b \notin V$, and so is open. Then in the convolution product the function we are integrating over at $a \in A$ is exactly the composite $\alpha \circ i_a : A \to \mathbb{C}$ which is Borel-measurable, being the composite of Borel-measurable functions, so we can consider its integral.

It remains to show that $||f * g||_{L^1} \leq ||f||_{L^1}||g||_{L^1} < \infty$ as this will also imply that the integral defining $f * g$ exists μ-almost everywhere. Note that by Lemma 2.10 $\text{supp}(f)$ and $\text{supp}(g)$ are σ-compact, and as μ is a Radon measure this also implies that they are σ-finite with respect to the measure. Note that if $K, L \subseteq A$ are compact, then $K \times L \subseteq A \times A$ is compact [Mun18, Thm. 26.7]. Observe that $KL = \{kl \in A \,|\, k \in K,\, l \in L\}$ is the continuous image of $K \times L$ under the multiplication map for A, so KL is also compact. As an immediate corollary this implies that the product of σ-compact subsets is σ-compact since a countable union of countable unions is a countable union.

From the observations in the previous paragraph we have that $B := \text{spec}(f) \times (\text{spec}(f)\text{spec}(g))$, which is the support of the $L^1(A \times A)$ function α described above, is σ-compact. Therefore we can apply Fubini's Theorem [Tay06, Thm. 6.4] in our computations. Explicitly, observe that

$$\begin{aligned}
\int_A |f * g(a)|d\mu(a) &= \int_{\text{spec}(f)\text{spec}(g)} |f * g(a)|d\mu(a) \\
&\leq \int_{\text{spec}(f)\text{spec}(g)} \int_{\text{spec}(f)} |f(b)g(b^{-1}a)|d\mu(b)d\mu(a) \\
&= \int_{\text{spec}(f)} \int_{\text{spec}(f)\text{spec}(g)} |g(b^{-1}a)|d\mu(a)|f(b)|d\mu(b) \quad \text{(by Fubini's Theorem)} \\
&= \int_{\text{spec}(f)} \int_A |g(b^{-1}a)|d\mu(a)|f(b)d\mu(b) \\
&\qquad \text{(since } g(b^{-1}a) = 0 \text{ for } b \in \text{spec}(f) \text{ if } a \notin \text{spec}(f)\text{spec}(g))
\end{aligned}$$

$$= \int_{\text{spec}(f)} \int_A |g(a)| d\mu(a) |f(b)| d\mu(b)$$

(by Proposition 2.7 applied to the inner integral)

$$= \int_A \int_A |g(a)| d\mu(a) |f(b) d\mu(b) = ||f||_{L^1} ||g||_{L^1}$$

completing the proof. ■

Note that since $L^1(A)$ is a Banach space [Tay06, Thm. 4.4], Proposition 3.1 implies that $L^1(A)$ is also a Banach algebra with the convolution product. In fact, since A is an abelian group, $L^1(A)$ with the convolution product is a commutative Banach algebra. First, for any $f, g \in L^1(A)$ and any $a \in A$ we can use Proposition 2.7 and the fact A is abelian to compute

$$f * g(a) = \int_A f(b) g(b^{-1}a) d\mu(b) = \int_A f(ba) g((ba)^{-1}a) d\mu(b) = \int_A f((b^{-1})^{-1}a) g(b^{-1}) d\mu(b)$$

Applying Corollary 2.11 to the final integral gives

$$f * g(a) = \int_A f((b^{-1})^{-1}a) g(b^{-1}) d\mu(b) = \int_A f(b^{-1}a) g(b) d\mu(b) = g * f(a)$$

so the convolution is indeed commutative.

In addition to a product structure, the Banach algebra $L^1(A)$ also comes equipped with a natural action by the dual group $\hat{A}$ of A, where the dual group of A is defined as follows.

Definition 3.2 [DE14, p. 61] The dual group of an abelian locally compact group A is the group

$$\hat{A} := \{\chi : A \to \mathbb{C} \,|\, \forall a \in A, |\chi(a)| = 1, \text{ and } \chi \text{ is a continuous group homomorphism}\}$$

where for $\chi, \psi \in \hat{A}$, $\chi \cdot \psi : A \to \mathbb{C}$, $\chi^{-1} : A \to \mathbb{C}$, and the identity $e : A \to \mathbb{C}$, are defined for all $a \in A$ by $(\chi \cdot \psi)(a) = \chi(a)\psi(a)$, $\chi^{-1}(a) = \frac{1}{\chi(a)}$, and $e(a) = 1$.

The dual group $\hat{A}$ acts on the Banach algebra $L^1(A)$ via evaluation of the Fourier transform. Explicitly, for $f \in L^1(A)$ we define the Fourier transform of f to be the map $\hat{f} : \hat{A} \to \mathbb{C}$ given by the integral

$$\hat{f}(\chi) := \int_A f(a) \chi(a^{-1}) d\mu(a), \quad \forall \chi \in \hat{A} \tag{3.1}$$

Note that for any $\chi \in \hat{A}$, since χ is normalized we can compute

$$|\hat{f}(\chi)| = \left| \int_A f(a) \chi(a^{-1}) d\mu(a) \right| \leq \int_A |f(a)| \cdot |\chi(a^{-1})| d\mu(a) = \int_A |f(a)| d\mu(a) = ||f||_{L^1} < \infty$$

where the last inequality holds since $f \in L^1(A)$, so the Fourier transform of L^1 functions is well-defined. Then for each $\chi \in \hat{A}$ we can define a map $\delta_\chi : L^1(A) \to \mathbb{C}$ given by

$$\delta_\chi(f) = \hat{f}(\chi), \quad \forall \chi \in \hat{A} \tag{3.2}$$

which is an algebra homomorphism. In order to show that δ_χ is an algebra homomorphism for each $\chi \in \hat{A}$, it is sufficient to prove the following lemma which shows that the Fourier transform is linear and that it turns convolution products into pointwise products.

Lemma 3.3 [DE14, Lem. 1.7.2] For $f, g \in L^1(A)$ and $\alpha \in \mathbb{C}$, $\widehat{f + \alpha g} = \hat{f} + \alpha\hat{g}$ and $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Proof. Let $f, g \in L^1(A)$ and $\alpha \in \mathbb{C}$. Then for $\chi \in \hat{A}$ we can use linearity of integration in L^1 [Tay06, Prop. 3.7] to compute

$$\begin{aligned}
\widehat{f + \alpha g}(\chi) &= \int_A (f + \alpha g)(a)\chi(a^{-1})d\mu(a) \\
&= \int_A f(a)\chi(a^{-1})d\mu(a) + \alpha \int_A g(a)\chi(a^{-1})d\mu(a) = \hat{f}(\chi) + \alpha\hat{g}(\chi)
\end{aligned}$$

so the Fourier transform is linear. Next, for the convolution, by the proof of Proposition 3.1 and Lemma 2.10, $f * g$ has σ-finite support so we can apply Fubini's Theorem [Tay06, Thm. 6.4] to compute

$$\begin{aligned}
\widehat{f * g}(\chi) &= \int_A (f * g)(a)\chi(a^{-1})d\mu(a) \\
&= \int_A \left[\int_A f(b)g(b^{-1}a)d\mu(b)\right] \chi(a^{-1})d\mu(a) \\
&= \int_A \left[\int_A f(b)g(b^{-1}a)\chi(b^{-1})\chi(ba^{-1})d\mu(b)\right] d\mu(a) \\
&\qquad \text{(since } \chi \text{ is a group homomorphism)} \\
&= \int_A \left[\int_A g(b^{-1}a)\chi((b^{-1}a)^{-1})d\mu(a)\right] f(b)\chi(b^{-1})d\mu(b) \\
&= \int_A \left[\int_A g(a)\chi(a^{-1})d\mu(a)\right] f(b)\chi(b^{-1})d\mu(b) \\
&\qquad \text{(applying Proposition 2.7 to the inner integral)} \\
&= \int_A f(b)\chi(b^{-1})\hat{g}(\chi)d\mu(b) \\
&= \hat{f}(\chi)\hat{g}(\chi)
\end{aligned}$$

which completes the proof. ■

Next, we aim to show that for every $\chi \in \hat{A}$, δ_χ is continuous and all non-zero continuous algebra homomorphisms $L^1(A) \to \mathbb{C}$ are of this form. This will allow us to leverage the theory of Banach algebras to conclude that $\hat{A}$ has a natural topological structure which makes it a locally compact abelian topological group.

For continuity of δ_χ, $\chi \in \hat{A}$, the fact that δ_χ is linear and $|\delta_\chi(f)| \leq ||f||_{L^1}$ for all $f \in L^1(A)$ implies that δ_χ is a bounded linear operator. Thus, from a well-known result in functional analysis we have that δ_χ is continuous [Con94, Prop. 1.1, p. 26]. It remains to show that any non-zero continuous algebra homomorphism $L^1(A) \to \mathbb{C}$ is of this form.

Proposition 3.4 [Fol94, Thm. 4.3] For all $\chi \in \hat{A}$, all non-zero continuous algebra homomorphisms are of the form δ_χ for some $\chi \in \hat{A}$.

Proof. Let $\alpha : L^1(A) \to \mathbb{C}$ be a non-zero continuous algebra homomorphism. Then in particular α is a continuous linear functional, so by the duality theorem for L^p spaces [Coh13, Prop. 3.5.5] there exists $\phi \in L^\infty(A)$ such that α is given by integration against ϕ. Now by definition

$\phi : A \to \mathbb{C}$ is a map which is bounded.

Consider $f, g \in L^1(A)$, where $\text{supp}(f)$ and $\text{supp}(g)$ are σ-compact by Lemma 2.10. Then in particular there union $B = \text{supp}(f) \cup \text{supp}(g)$ is σ-compact, and hence σ-finite. Thus, integrating f and g over A or B will give the same result since f and g are zero outside of B. It follows that we can apply Fubini's Theorem [Tay06, Thm. 6.4] along with the work in the proof of Proposition 3.1. Observe that since α is an algebra homomorphism

$$\begin{aligned}
\int_A \alpha(f)\phi(a)g(a)d\mu(a) &= \alpha(g)\alpha(f) = \alpha(g * f) \\
&= \int_A \phi(a)(g * f)(a)d\mu(a) \\
&= \int_{\text{spec}(g)\text{spec}(f)} \phi(a)(g * f)(a)d\mu(a) \\
&= \int_{\text{spec}(g)\text{spec}(f)} \int_{\text{spec}(g)} \phi(a)f(b^{-1}a)g(b)d\mu(b)d\mu(a) \\
&\qquad \text{(using Fubini's Theorem)} \\
&= \int_{\text{spec}(g)} \int_{\text{spec}(g)\text{spec}(f)} \phi(a)f(b^{-1}a)d\mu(a)g(b)d\mu(b) \\
&\qquad \text{(using Fubini's Theorem)} \\
&= \int_{\text{spec}(g)} \alpha(f \circ l_{b^{-1}})g(b)d\mu(b) \\
&= \int_A \alpha(f \circ l_{b^{-1}})g(b)d\mu(b)
\end{aligned}$$

As the linear functional is uniquely represented by the function in L^1 being integrated against, we must have that $\alpha(f)\phi(a) = \alpha(f \circ l_{a^{-1}})$, μ-almost everywhere. Since α is non-zero we have $f \in L^1(A)$ such that $\alpha(f) \neq 0$, so as $\phi(a)$ was independent of f in the above expression we can define $\phi(a) := \frac{\alpha(f \circ l_{a^{-1}})}{\alpha(f)}$ for all $a \in A$. Then since α is continuous ϕ will be continuous.

Finally, the above computation also implies that for any $a, b \in A$,

$$\phi(ab)\alpha(f) = \alpha(f \circ l_{(ab)^{-1}}) = \alpha(f \circ l_{b^{-1}} \circ l_{a^{-1}}) = \phi(a)\alpha(f \circ l_{b^{-1}}) = \phi(a)\phi(b)\alpha(f)$$

so dividing by $\alpha(f)$ implies ϕ is multiplicative.

Applying this result inductively implies that $\phi(a)^n = \phi(a^n)$ for any $a \in A$ and any $n \in \mathbb{Z}$, so in order for ϕ to be bounded we must have that in fact $|\phi(a)| = 1$ for all $a \in A$. This proves that $\phi \in \hat{A}$, and so $\alpha = \delta_{\phi^{-1}}$, where ϕ^{-1} is the inverse of ϕ in the group $\hat{A}$. ■

Proposition 3.4 implies that the dual group $\hat{A}$ can be identified with what is known as the spectrum of the commutative Banach algebra $L^1(A)$ [Fol94, p. 15].[1] This identification gives a natural topology on $\hat{A}$ which makes it a locally compact abelian topological group. We can describe this topology in three equivalent forms, which we state in the below proposition, though we will not prove this proposition as the proof goes beyond the scope of this paper. The proof of the equivalence of (1) and (2) is given in [Mun18, Thm. 46.8] while the proof of the equivalence for (2) and (3) is given in [Fol94, Thm. 3.31].

[1] The spectral theory of Banach and C^*-algebras is beyond the scope of the current paper, so we refer the interested reader to [Fol94].

Proposition 3.5 Let A be a locally compact topological group which is abelian. Then the dual group, $\hat{A}$, is also a locally compact topological group with any of the following three equivalent topologies.

(1) The compact-open topology on $\hat{A}$ which is given by the subbasis of sets

$$S(K, U) = \{\chi \in \hat{A} \,|\, \chi(K) \subseteq U\}$$

for all $K \subseteq A$ compact and $U \subseteq \mathbb{C}$ open.

(2) The topology of compact convergence on $\hat{A}$, which is given by the basic open balls

$$N(\phi; \epsilon, K) = \{\psi \in \hat{A} \,|\, \sup_{x \in K} |\psi(x) - \phi(x)| < \epsilon\}$$

for all $\phi \in \hat{A}$, $\epsilon > 0$, and $K \subseteq A$ compact.

(3) The weak-* topology on $\hat{A}$, which is the topology of pointwise convergence. In other words, it is the coarsest topology on $\hat{A}$ such that for each $a \in A$, the evaluation map $\text{ev}_a : \hat{A} \to \mathbb{C}$ given by $\text{ev}_a(\chi) = \chi(a)$ for $\chi \in \hat{A}$ is continuous.

It is an important consequence of the theory of C^*-algebras that given this topological structure on the dual group $\hat{A}$, the Fourier transform of L^1 functions form a dense subspace of the space $C_0(\hat{A})$ of continuous functions which vanish at infinity [Fol94, Prop. 4.14]. In the current context a continuous function $f : \hat{A} \to \mathbb{C}$ vanishing at infinity means that for any $\epsilon > 0$, there exists a compact set $K \subseteq \hat{A}$ for which $|f(\chi)| < \epsilon$ when $\chi \in \hat{A} \backslash K$.

Using the three equivalent descriptions of the topology on the dual group $\hat{A}$ we can obtain a powerful relation between the topology on A and that on $\hat{A}$.

Proposition 3.6 [DE14, Prop. 3.1.5] Let A be a locally compact abelian topological group. Then A being compact implies that $\hat{A}$ has the discrete topology, and A having the discrete topology implies $\hat{A}$ is compact.

Proof. To begin suppose A is compact. Consider the open ball of radius $1/2$ at 1 in $\mathbb{C}$, $B_{1/2}(1)$. Then by topology (1) in Proposition 3.5 we have the basic open set $S(A, B_{1/2}(1))$ in $\hat{A}$ as A is compact. Now, towards a contradiction suppose we have $\chi \in S(A, B_{1/2}(1))$ which is not the unit of $\hat{A}$. Then there exists $a \in A$ such that $\chi(a) \neq 1$, so $\chi(a) = e^{i\theta}$ for some $\theta \in (0, 2\pi)$. Then there exists $n \in \mathbb{Z}$ such that $n\theta \in (\pi/2 + k, 3\pi/2 + k)$ for some integer k. But then since χ is a group homomorphism $\chi(a^n) = e^{in\theta} \notin B_{1/2}(1)$, contrary to the assumption that $\chi \in S(A, B_{1/2}(1))$. Thus we must have that $S(A, B_{1/2}(1))$ is the singleton containing the identity element of $\hat{A}$. But then for any $\chi \in \hat{A}$, $\chi S(A, B_{1/2}(1))$ is the singleton containing χ, so $\hat{A}$ has the discrete topology.

Conversely, suppose A has the discrete topology, so the only compact subsets of A are the finite subsets. Since A is discrete, a similar argument to that in Example 2.1 implies that the counting measure is a Haar measure on A. Then for any $f \in L^1(A)$, if e denotes the identity element of A, we have that $f * \chi_{\{e\}}(a) \int_A f(b)\chi_{\{e\}}(b^{-1}a)d\mu(b) = f(a)$. It follows that $\chi_{\{e\}}$ is a multiplicative identity for $L^1(A)$. From the theory of unital Banach algebras and our identification of $\hat{A}$ with the spectrum of $L^1(A)$ it follows that $\hat{A}$ is compact [Fol94, p. 6]. ■

We will now move on to demonstrating how Pontryagin duality implies the Fourier inversion theorem. We will also prove that Pontryagin duality gives a powerful converse to Proposition 3.6.

3.2 Pontryagin Duality and the Fourier Transform

For an abelian locally compact group A, we have a natural continuous group homomorphism from A to its double dual, $\hat{\hat{A}}$, given by the map $\mathrm{ev} : A \to \hat{\hat{A}}$ which sends an element $a \in A$ to the evaluation map $\mathrm{ev}_a : \hat{A} \to \mathbb{C}$ [DE14, Sec. 3.5]. Note that this is a well-defined map since for each $a \in A$ the map ev_a is continuous by the weak-* description of the topology on $\hat{A}$, and ev_a is a group homomorphism since for any $\chi, \psi \in \hat{A}$,

$$\mathrm{ev}_a(\chi \cdot \psi) = (\chi \cdot \psi)(a) = \chi(a)\psi(a) = \mathrm{ev}_a(\chi)\mathrm{ev}_a(\psi)$$

Further, as $|\chi(a)| = 1$ for all $\chi \in \hat{A}$ and $a \in A$, we also have that $|\mathrm{ev}_a(\chi)| = |\chi(a)| = 1$.

In addition to ev_a being a continuous group homomorphism for each $a \in A$, the map ev itself is a group homomorphism since for any $a, b \in A$ and $\chi \in \hat{A}$

$$(\mathrm{ev}_a \cdot \mathrm{ev}_b)(\chi) = \chi(a)\chi(b) = \chi(ab) = \mathrm{ev}_{ab}(\chi)$$

The content of Pontryagin duality is that ev, which is sometimes called the Pontryagin map [DE14, p. 74], is an isomorphism of topological groups. The primary goal of this section is to develop sufficient theory to demonstrate that this duality allows us to quickly prove the Fourier inversion formula for L^1 functions with L^1 Fourier transform.

In order to reach this point we will need some preliminary results on a special class of $L^1(A)$ functions, including a Fourier inversion result for these functions. The functions we wish to consider are those $f \in L^1(A)$ such that there exists a complex Radon measure λ on $\hat{A}$ making the equality

$$f(a) = \int_{\hat{A}} \mathrm{ev}_a(\chi) d\lambda(\chi)$$

hold for any $a \in A$. As in [Fol94, p. 105] we denote the subspace of these functions in $L^1(A)$ by $\mathcal{B}^1(A)$.

The space of functions $\mathcal{B}^1(A)$ has two important properties which will be valuable for the proof of Fourier inversion on $\mathcal{B}^1(A)$ functions. We will not prove these properties here for the sake of space, and as they are beyond the scope of the current paper. The first property is that if $K \subseteq \hat{A}$ is compact subset, then there exists a continuous and compactly supported element $f \in \mathcal{B}^1(A)$ such that its Fourier transform $\hat{f}$ is non-negative and is positive on K [Fol94, Lem. 4.20]. The second important property is that if $f, g \in \mathcal{B}^1(A)$ are determined by Radon measures λ and ρ, then integrating with respect to $\hat{f} d\rho$ is equivalent to integrating with respect to $\hat{g} d\lambda$ [Fol94, Lem. 4.21].

Using these properties of $\mathcal{B}^1(A)$ we can prove a first case of the Fourier inversion formula for $\mathcal{B}^1(A)$ functions [Fol94, Thm. 4.22]. The full proof of [Fol94, Thm. 4.22] is beyond the scope of this paper, so we will provide a brief sketch of the main ideas.

Theorem 3.7 [Fol94, Thm. 4.22] If $f \in \mathcal{B}^1(A)$, then $\hat{f} \in L^1(\hat{A})$ and there exists a Haar measure ξ on $\hat{A}$ such that for any $a \in A$

$$f(a) = \int_{\hat{A}} \mathrm{ev}_a(\chi) \hat{f}(\chi) d\xi(\chi)$$

Proof Idea/Sketch. Since $\hat{A}$ is a locally compact space we can use the Riesz-representation Theorem [Rud87, Thm. 2.14] which states that we have a correspondence between positive bounded linear functionals on $C_{00}(\hat{A})$ and Radon measures on $\hat{A}$. This Riesz-representation

Theorem allows us to construct our desired Haar measure by instead constructing a bounded linear functional on $C_{00}(\hat{A})$ which is translation invariant.

Let $\varphi \in C_{00}(\hat{A})$. By the properties discussed above we have an $f \in C_{00}(A) \cap \mathcal{B}^1(A)$ such that $\hat{f} \geq 0$ and $\hat{f} > 0$ on $\mathrm{supp}(\varphi)$ since $\mathrm{supp}(\varphi)$ is compact. Let λ be the Radon measure associated with $f \in \mathcal{B}^1(A)$. Then we can integrate φ with respect to $\frac{1}{\hat{f}} d\lambda$ since $\hat{f} > 0$ on $\mathrm{supp}(\varphi)$. Let $\Phi(\varphi)$ denote this integral.

The second property given above the theorem implies that $\Phi(\varphi)$ is independent of the element of $C_{00}(A) \cap \mathcal{B}^1(A)$ chosen to satisfy the given positivity requirements. Then if $\varphi, \psi \in C_{00}(\hat{A})$ and $\alpha \in \mathbb{C}$, since the finite union of compact sets is compact we can find $f \in C_{00}(A) \cap \mathcal{B}^1(A)$ such that $\hat{f} \geq 0$ and $\hat{f} > 0$ on $\mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi)$. It follows by linearity of integration that

$$\Phi(\varphi + \alpha\psi) = \int_{\mathrm{supp}(\varphi)\cup\mathrm{supp}(\psi)} (\varphi + \alpha\psi)\frac{1}{\hat{f}} d\lambda = \int_{\mathrm{supp}(\varphi)} \varphi\frac{1}{\hat{f}} d\lambda + \alpha \int_{\mathrm{supp}(\psi)} \psi\frac{1}{\hat{f}} d\lambda = \Phi(\varphi) + \alpha\Phi(\psi)$$

Thus Φ is a linear functional on $C_{00}(\hat{A})$.

Finally, to sketch the translational invariance of Φ consider $\varphi \in C_{00}(\hat{A})$ and $\xi \in \hat{A}$. Then let $f \in C_{00}(A) \cap \mathcal{B}^1(A)$ with associated Radon measure λ such that $\hat{f} \geq 0$ and $\hat{f} > 0$ on the compact set $\mathrm{supp}(\varphi) \cup \mathrm{supp}(\varphi \circ l_\xi)$. Note that for any $a \in A$ we can compute

$$\xi(a)f(a) = \int_{\hat{A}} \xi(a)\mathrm{ev}_a(\chi)d\lambda(\chi) = \int_{\hat{A}} \mathrm{ev}_a(\xi\chi)d\lambda(\chi) = \int_{\hat{A}} \mathrm{ev}_a(\chi)d\lambda(\xi^{-1}\chi)$$

where we performed a similar change of variables to that in Proposition 2.7, but since λ is not necessarily a Haar measure $d\lambda(\xi^{-1}\chi)$ does not equal $d\lambda(\chi)$ in general. Note that this implies $\xi f \in C_{00}(A) \cap \mathcal{B}^1(A)$ with associated Radon measure given by the measure associated with the change of variables in $d\lambda(\xi^{-1}\chi)$, which can be realized explicitly as the composition of λ with the action of $l_{\xi^{-1}}$ on the Borel measurable sets in $\hat{A}$.

Note that we can describe the Fourier transform of ξf by

$$\widehat{(\xi f)}(\chi) = \int_A f(a)\xi(a)\chi(a^{-1})d\mu(a) = \int_A f(a)(\xi^{-1} \cdot \chi)(a^{-1})d\mu(a) = \hat{f}(\xi^{-1}\chi)$$

using the definitions of multiplication and inversion in $\hat{A}$. Performing the same kind of change of variables trick again and the independence of Φ on the $C_{00}(A) \cap \mathcal{B}^1(A)$ function, we can compute

$$\Phi(\varphi \circ l_\xi) = \int_{\hat{A}} \varphi(\xi\chi)\frac{1}{\hat{f}(\chi)}d\lambda(\chi) = \int_{\hat{A}} \varphi(\chi)\frac{1}{\hat{f}(\xi^{-1}\chi)}d\lambda(\xi^{-1}\chi) = \int_{\hat{A}} \varphi(\chi)\frac{1}{\widehat{(\xi f)}(\chi)}d\lambda(\xi^{-1}\chi) = \Phi(\varphi)$$

as desired. This completes our sketch of the important points of the proof. ■

We now have sufficient tools to state Pontryagin duality for abelian locally compact groups and show how it implies the full Fourier inversion formula. Note that since $\mathrm{ev} : A \to \hat{\hat{A}}$ has already been proven to be a group homomorphism in the discussion at the start of the section, it remains only to show that it is a homeomorphism [Fol94, Thm. 4.32]. As the proofs of Pontryagin duality in the literature, such as in [Fol94, Thm. 4.32] and [DE14, Thm. 3.5.5] use techniques in the theory of Banach and C^*-algebras which are beyond the scope of this paper, we only state Pontryagin duality so as to focus on its implications.

Theorem 3.8 The map $\mathrm{ev} : A \to \hat{\hat{A}}$ is a group isomorphism which is also a homeomorphism.

Pontryagin duality comes with a myriad of important and valuable applications. For example, an immediate consequence of Pontryagin duality and Proposition 3.6 is that a locally compact abelian topological group A is compact, respectively discrete, if and only if its dual group $\hat{A}$ is discrete, respectively compact. To illuminate this result, let us return to the discrete lattice group $\mathbb{Z}^n$.

Example 3.1 (Dual group of $\mathbb{Z}^n$) :

Since $\mathbb{Z}^n$ is a discrete group, Proposition 3.6 implies that its dual group is compact. In fact, $\widehat{\mathbb{Z}^n}$ is equivalent to the n-torus group $(S^1)^n$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the circle group under complex multiplication [Fol94, Thm. 4.6]. Explicitly, an element $(\alpha_1, ..., \alpha_n) \in (S^1)^n$ acts on an element $(m_1, ..., m_n) \in \mathbb{Z}^n$ by exponentiation and multiplication

$$(\alpha_1, ..., \alpha_n) \star (m_1, ..., m_n) := \alpha_1^{m_1} \cdots \alpha_n^{m_n}$$

where we use $\star$ to denote the action of an element of $(S^1)^n$ on an element of $\mathbb{Z}^n$.

In addition to this topological result, Pontryagin duality allows us to upgrade the inversion result in Theorem 3.8 to all $L^1(A)$ functions that land in $L^1(\hat{A})$.

Theorem 3.9 [Fol94, Thm. 4.33] If $f \in L^1(A)$ such that $\hat{f} \in L^1(\hat{A})$, then for μ-almost all $a \in A$ we have the equality

$$f(a) = \int_{\hat{A}} \mathrm{ev}_a(\chi)\hat{f}(\chi)d\xi(\chi)$$

for the Haar measure ξ from Theorem 3.2.

Proof. Let $f \in L^1(A)$ such that $\hat{f} \in L^1(\hat{A})$. Using the definition of Fourier transform and Corollary 2.11, we have that for $\chi \in \hat{A}$

$$\hat{f}(\chi) = \int_A \chi(a^{-1})f(a)d\mu(a) = \int_A \chi(a)f(a^{-1})d\mu(a)$$

Note that since μ is a Radon measure and f is L^1, the linear functional defined by integration with respect to $f(a^{-1})d\mu(a)$ determines a complex Radon measure by the Riesz-representation Theorem [Rud87, Thm. 6.19]. Thus, $\hat{f} \in \mathcal{B}^1(\hat{A})$, so by Theorem 3.7 and the uniqueness of the Riesz-representation Theorem [Rud87, Thm. 6.19], we must have that $\hat{\hat{f}}(a) = f(a^{-1})$ for μ-almost all $a \in A$, where we are identifying $\hat{\hat{A}}$ with A via the Pontryagin map and Theorem 3.8. This completes the proof. ∎

The Fourier inversion formula in Theorem 3.9 is an incredibly powerful result with a number of widespread application. In the case of our example where $A = \mathbb{Z}$ and $\hat{A} = S^1$, Theorem 3.9 applied to a function $f \in L^1(A)$ such that $\hat{f} \in L^1(\hat{A})$ gives the expansion of f

$$f(m) = \int_0^{2\pi} e^{2\pi i m\theta}\hat{f}(e^{2\pi i\theta})d\theta$$

where $\hat{f}(e^{2\pi i\theta}) = \sum_{n\in\mathbb{Z}} f(n)e^{-2\pi i n\theta}$ is often called the discrete time-series Fourier transform of

f. This type of Fourier transform, along with its inversion formula, is essential in fields such as physics for the analysis and filtering of discrete time-series data that is subjected to external noise.

4 Conclusion

In this paper we developed and investigated the Fourier transform on locally compact abelian groups, emphasizing the role Pontryagin duality plays in obtaining essential inversion formulas for the Fourier transform. In this process we have introduced locally compact topological groups and their Haar measures, which allow for invariant integration on such groups. We also investigated the Banach algebra structure on $L^1(A)$ induced by the convolution product when A is a locally compact abelian group. Additionally, we demonstrated that $L^1(A)$ comes equipped with a natural action by the dual group $\hat{A}$ of A, which induces a locally compact group structure on $\hat{A}$. Finally we concluded with a number of proofs and sketches which lead to the proof that Pontryagin duality for locally compact abelian groups implies the Fourier inversion formula for L^1 functions with L^1 fourier transforms. Throughout the paper we motivated important concepts with the accessible example of the discrete group $\mathbb{Z}^n$, showing in the end that the Fourier transform on $\mathbb{Z}$, where $n = 1$, produces the discrete time-series Fourier transform, a variety of the Fourier transform which has a host of uses in data-driven fields.

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