Local to Global: An Introduction to Sheaves

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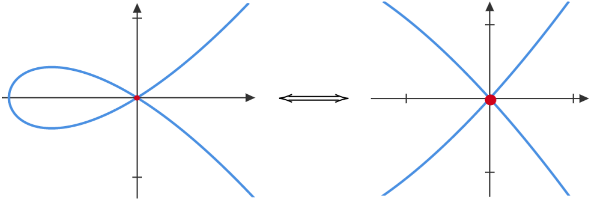
Math 511 Presentation

Motivation

Motivating Question

How can we study the relation between local and global properties of geometric spaces algebraically?



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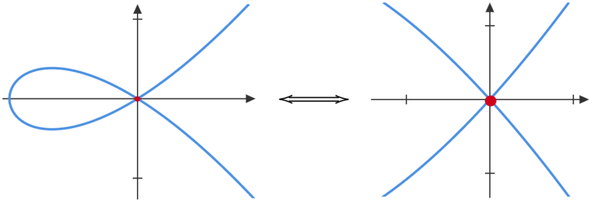

One Answer: Sheaves and sheaf cohomology!

What is a sheaf?

- Throughout let $(X, \tau) \in$ **Top**.

Def^n: (Sheaves)

A **pre-sheaf** on X with values in $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{O}(X)^{op} \to \mathcal{C}$$

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$$\mathcal{F} : \mathcal{O}(X)^{op} \to \mathcal{C}$$

If $\forall U \in \mathcal{O}(X)$ $\mathcal{F}$ satisfies

- $\forall U = \bigcup_{i \in I} U_i, \forall s_i \in \mathcal{F}(U_i)$,

$$\forall i, j \in I(s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}) \implies \exists! s \in \mathcal{F}(U), \ \forall i \in I(s|_{U_i} = s_i)$$

it is called a **sheaf**

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Example: Smooth Manifolds

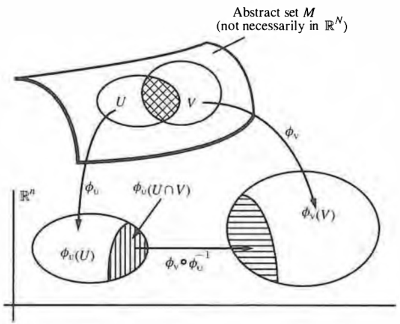
Eg: Smooth Manifolds

A smooth manifold is a pair $(M, \mathcal{O}_M)$, with $M \in$ **Top** and $\forall U \in M$, $\mathcal{O}_M(U) =$ smooth real-valued functions, satisfying

- $\forall p \in M, \exists U, p \in U$, such that

$$(U, \mathcal{O}_M|_U) \cong (\mathbb{R}^n, \mathcal{O}_{C^\infty})$$

for some $n \in \mathbb{N}$



Maps of sheaves

Defⁿ: (Sheaf Map)

A map between sheaves $\mathcal{F}, \mathcal{G} : \mathcal{O}(X)^{op} \to \mathcal{C}$ is a collection

$$(\eta_U \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(U), \mathcal{G}(U)))_{U \in \mathcal{O}(X)}$$

such that the diagram commutes for any $U \subseteq V \in \mathcal{O}(X)$.

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\;|_U\;} & \mathcal{F}(U) \\ {\scriptstyle \eta_V}\downarrow & & \downarrow{\scriptstyle \eta_U} \\ \mathcal{G}(V) & \xrightarrow[\;|_U\;]{} & \mathcal{G}(U) \end{array}$$

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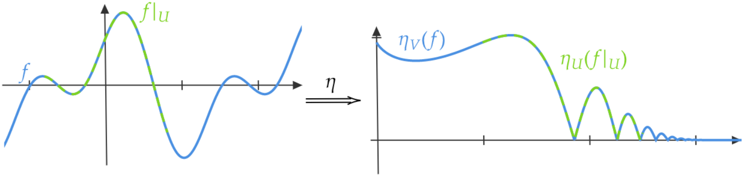

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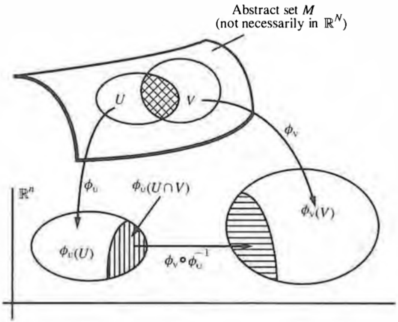

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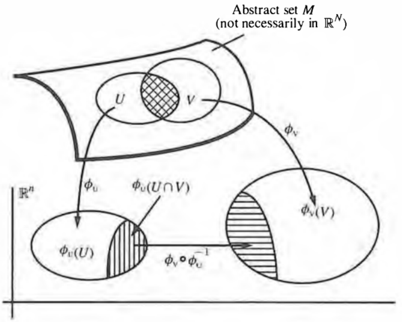

Observation: Differentiation and other operations on functions depend only on local behaviour

Characterizing Locality Through Universality: Stalks

- Fix a sheaf $\mathcal{F} : \mathcal{O}(X)^{op} \to \mathcal{C}$

Def^{n}: (Stalks)

The **stalk** of $\mathcal{F}$ at $x \in X$ is **colimit**

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

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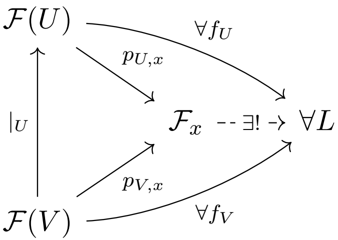

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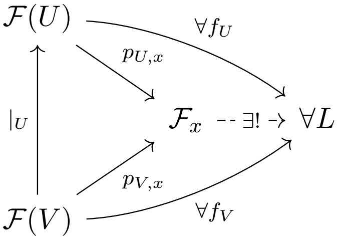

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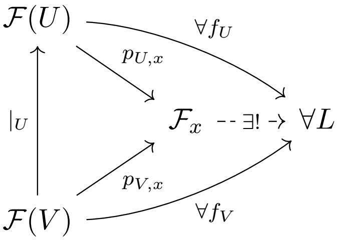
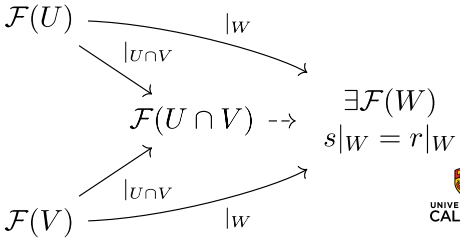


$$(s \in \mathcal{F}(V)) \sim (r \in \mathcal{F}(U)) \implies$$



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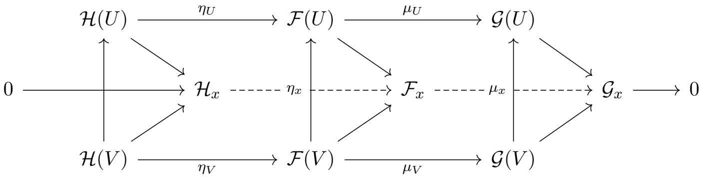
Exact

A sequence of sheaves on X, $0 \to \mathcal{H} \xrightarrow{\eta} \mathcal{F} \xrightarrow{\mu} \mathcal{G} \to 0$, induces a sequence



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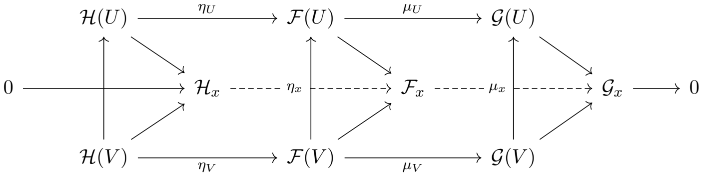

Remark. The original sequence is exact if and only if

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is exact $\forall x \in X$.

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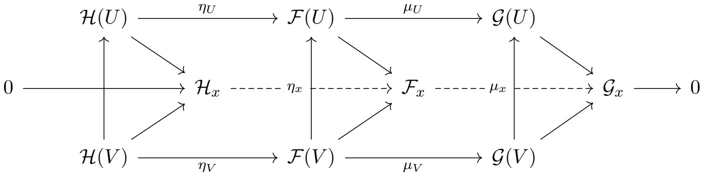

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Note

Surjectivity is local!

The Structure of Geometric Spaces: Category of Ringed Spaces

Def^n: (Ringed Space)

A **ringed space** is a pair $(X, \mathcal{O}_X)$ of $X \in$ **Top**, and $\mathcal{O}_X : \mathcal{O}(X)^{op} \to$ **Ring** a sheaf of rings

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A map of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair of maps $\varphi : X \to Y$ and $\varphi^{\#} : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$

Inducing Sheaves

- Fix a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X over $\mathcal{C}$

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Def^n: (Push-forward)

The push-forward of $\mathcal{F}$ along f is the pre-sheaf

$$f_*\mathcal{F} : \mathcal{O}(Y)^{op} \to \mathcal{C}$$

given by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$

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Example: Smooth Manifolds Revisited

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- Let $(M, \mathcal{O}_M)$ be a smooth manifold
- Let $TM = \coprod_{p \in M} T_pM$ denote the tangent bundle
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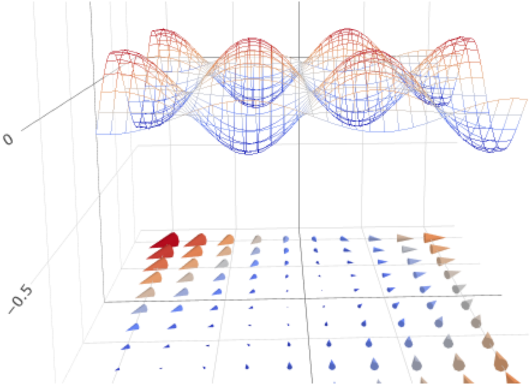

Global sections functor

- The **global sections functor**, $\Gamma : \mathcal{O}_X$-**Mod** $\to \mathcal{O}_X(X)$-**Mod**, is given by $\Gamma(\mathcal{F}) = \mathcal{F}(X)$

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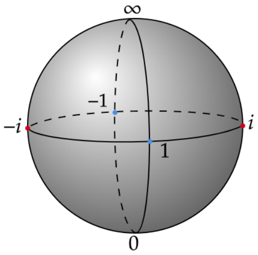

Figure: Riemann Sphere

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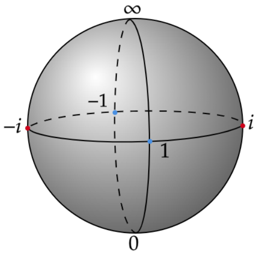
- By Liouville's Theorem $\mathcal{A}(X)$ consists of all constant functions



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Prop

Γ is a left-exact functor

Proof Idea: Let $0 \to \mathcal{H} \xrightarrow{\eta} \mathcal{F} \xrightarrow{\mu} \mathcal{G} \to 0$ be a SES. This induces a diagram

$$\begin{array}{ccccccccc} & & \Gamma(\mathcal{H}) & \xrightarrow{\eta_X} & \Gamma(\mathcal{F}) & \xrightarrow{\mu_X} & \Gamma(\mathcal{G}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H}_x & \overset{\eta_x}{\dashrightarrow} & \mathcal{F}_x & \overset{\mu_x}{\dashrightarrow} & \mathcal{G}_x & \longrightarrow & 0 \end{array}$$

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Sheaf Cohomology

Remark. We want to measure the failure of Γ to be right-exact.

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Construction. To extend Γ, for each $\mathcal{F} \in \mathcal{O}_X$-**Mod** we "take an injective resolution" $0 \to \mathcal{F} \to \mathcal{I}_\bullet$ and set

$$R^n\Gamma(\mathcal{F}) = H^n(\Gamma(\mathcal{I}_\bullet))$$

for $n \in \mathbb{Z}_{\geq 0}$.

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Question. Does every $\mathcal{F} \in \mathcal{O}_X$-**Mod** have an injective resolution?

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Existence of Sheaf Cohomology

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The category $\mathcal{O}_X$-**Mod** has enough injectives.

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Proof Sketch. Fix $\mathcal{F} \in \mathcal{O}_X$-**Mod**.

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- Define $\mathcal{I} : \mathcal{O}(X)^{op} \to$ **Ab** by $\mathcal{I}(U) = \prod_{x \in U} \mathcal{I}(x)$

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- It can be shown $\mathcal{I} \in \mathcal{O}_X$-**Mod** is injective, and the induced map $\iota : \mathcal{F} \hookrightarrow \mathcal{I}$ is a monomorphism

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Applications

Cor

A SES of $\mathcal{O}_X$-modules, $0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$, induces a long-exact sequence

$$0 \longrightarrow \Gamma(\mathcal{F}) \longrightarrow \Gamma(\mathcal{H}) \longrightarrow \Gamma(\mathcal{G}) \xrightarrow{\delta^0}$$

$$R^1\Gamma(\mathcal{F}) \longrightarrow R^1\Gamma(\mathcal{H}) \longrightarrow R^1\Gamma(\mathcal{G}) \dashrightarrow$$

$$R^n\Gamma(\mathcal{F}) \longrightarrow R^n\Gamma(\mathcal{H}) \longrightarrow R^n\Gamma(\mathcal{G})$$

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Canonical Example:

- Studying global properties of the complex logarithm

Thank you

Thank you for your time!

Any questions?

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References I

[1] P. Achar. *Perverse Sheaves and Applications to Representation Theory*. Mathematical Surveys and Monographs. American Mathematical Society, 2021. ISBN: 9781470455972.

[2] U. Bruzzo and B. Otero. *Derived Functors and Sheaf Cohomology*. Contemporary Mathematics and Its Applications: Monographs, E Series. World Scientific, 2020. ISBN: 9789811207280.

[3] D. Eisenbud and J. Harris. *The Geometry of Schemes*. Graduate Texts in Mathematics. Springer, 2000. ISBN: 9780387986388.

[4] S. Gelfand and Y. Manin. *Methods of Homological Algebra*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2002. ISBN: 9783540435839.

[5] nLab authors. *manifold*. `https://ncatlab.org/nlab/show/manifold`. Revision 49. Nov. 2022.

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