A Concrete Construction of Polynomial Functors in Chain Complexes

E Ea Thompson (UCID: 30087678)
Supervisor: Dr. Kristine Bauer

Abstract

A fundamental tool in algebraic topology is the use of algebraic objects to classify spaces up to a suitable notion of equivalence, namely homotopy equivalence. These algebraic objects are designed to be invariant under changes in a space up to equivalence, and can be described formally in the language of functors. In the modern literature, invariants themselves have rich and complicated structures, and are of mathematical interest in their own right. In this thesis we introduce a method for approximating invariants by simpler objects, known as polynomial functors, in analogy with the use of Taylor polynomials in analysis. Historically these polynomial functors have been defined only up to weak equivalence, as in [Goo03]. We provide an explicit construction of these polynomial functors, in line with [JM04] and [BJO$^+$18], in the case when our invariants are valued in chain complexes. Further, using techniques from homological algebra and category theory we show that our constructed polynomial functors satisfy a global property as universal approximations, extending the universality properties proved in [JM04] and [BJO$^+$18].

1 Introduction

Since the origin of the field, topologists have been concerned with how to analyze and distinguish topological spaces. Following observations by Emmy Noether in the early 1900s [Wei99], it was found that spaces can be studied using algebraic objects called invariants, such as certain chain complexes of abelian groups.

These observations transform the classification of topological spaces into the study of algebraic invariants. However, the study of invariants has itself evolved into an incredibly rich field with a number of powerful objects such as de Rham Cohomology for smooth manifolds, Hochschild Homology for associative algebras, and even Topological K-Theory which studies vector bundles on topological spaces [Wei99]. From the early 1990s through the early 2000s, Thomas Goodwillie began the development of a theory that approximates invariants through towers of simpler invariants called "polynomial functors", strongly in analogy with Taylor series in calculus [Goo90, Goo92, Goo03]. At about the same time, Johnson and McCarthy performed an analogous construction for invariants valued in chain complexes such that the resulting approximations are universal up to a notion of weak equivalence called quasi-isomorphism [JM04]. Bauer et al. built on this construction in [BJO$^+$18], providing explicit models for the polynomial functors and showing that they satisfy a stronger universality property in terms of pointwise chain homotopies.

In this paper we will perform the constructions of [JM04] and [BJO$^+$18] in terms of chain complexes valued in modules over a ring. Through this process we will demonstrate that the constructions in each paper agree up to **natural** chain homotopy equivalence. Following this initial construction and comparison we will prove the primary result of the thesis, Theorem 3.11. Theorem 3.11 establishes a universal property for polynomial functors which holds up to **natural** chain homotopy equivalence. This is stronger than the result in [JM04] (which holds up to quasi-isomorphism) or [BJO$^+$18] (which holds up to pointwise chain homotopy equivalence). Finally, to help illustrate the construction, the first degree polynomial functors for two simple operations appearing in linear algebra will be computed.

Acknowledgements: I would like to acknowledge and express my deep gratitude for the support of Dr. Kristine Bauer, Dr. Jason Parker, and PhD student Florian Schwarz, who provided a number of valuable insights throughout the duration of the project. I would also like to thank Dr. Balehowsky and Dr. Rios for their assistance in the formatting and structuring of my thesis and presentation.

2 Background Information

2.1 Categorical Preliminaries for Invariants of Spaces

In order to rigorously define and study algebraic invariants we require the notion of categories and functors. A category can be informally thought of as a universe consisting of similar mathematical objects with functions between them that preserve their structure. Although it is not the original reference, we will primarily refer to [Rie17] for our categorical background.

Definition 2.1 [Rie17, Defn. 1.1.1] A category $\mathcal{C}$ consists of the following data:

- A class of mathematical objects, $\mathsf{Ob}(\mathcal{C})$
- For any pair of objects, $A, B \in \mathsf{Ob}(\mathcal{C})$, a set of maps, or arrows, $\mathcal{C}(A, B)$
- For every object $A \in \mathsf{Ob}(\mathcal{C})$, a chosen map $1_A \in \mathcal{C}(A, A)$
- For every pair of maps $f : A \to B$ and $g : B \to C$, a composite map $g \circ f : A \to C$

where composition is associative and for $f : A \to B$, $f \circ 1_A = f = 1_B \circ f$.

A map $f : A \to B$ is said to be an *isomorphism* if there exists an inverse map $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

For our study the primary categories to keep in mind are the category Top_* of topological spaces with base points and continuous maps that preserve base points, the category Ab of abelian groups and group homomorphisms, and the category R-Mod of modules over a ring R with R-module maps. Here an R-module is an abelian group M with a scalar multiplication operation $R \times M \to M$ satisfying the same axioms as that for vector spaces. A map of R-modules $f : M \to N$ is then a map of abelian groups which is linear with respect to the scaling operation from R.

In [BJO$^+$18], a generalization of the category Ab, known as abelian categories, is used. However, the arguments we will use in this thesis will generalize to that case using the Freyd-Mitchell Embedding Theorem which says that any abelian category can be embedded into a category of R-modules for some ring R [Wei94, Thm 1.6.1]. This will allow us to work in a suitable level of generality while making the proofs accessible to a wider audience.

Next, as with the objects we consider inside of categories, we want a way to map between categories. Such a map should send all the data in one category to appropriate data in another category while preserving the conditions on that data. This brings us to the definition of a functor.

Definition 2.2 [Rie17, Defn. 1.3.1] A functor $F : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map of objects, $F : \mathsf{Ob}(\mathcal{A}) \to \mathsf{Ob}(\mathcal{B})$, and for every pair of objects, $A, A' \in \mathsf{Ob}(\mathcal{A})$, a map of arrows:

$$F_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$$

The arrow maps are required to preserve identities and composition. That is, $F_{A,A}(1_A) = 1_{F(A)}$ and for $A \xrightarrow{f} A' \xrightarrow{g} A''$, $F_{A,A''}(g \circ f) = F_{A',A''}(g) \circ F_{A,A'}(f)$.

For simplicity of notation we will often hide the subscripts when talking about functors applied to maps. With categories and functors defined we are able to understand the initial statement of our problem; how do we simplify and study algebraic invariants which depend functorially on the space being considered? Explicitly, functorial algebraic invariants of spaces are functors $F : \mathsf{Top}_* \to \mathcal{C}$ which send homotopy equivalences, or other appropriate weak equivalences of spaces, to suitable equivalences in the category $\mathcal{C}$, where $\mathcal{C}$ is a category of algebraic objects like Ab. A prime example of functorial invariants are the higher fundamental groups, which are described by a family of functors $\pi_n : \mathsf{Top}_* \to \mathsf{Ab}$, for $n \geq 2$ (see e.g. [Hat02, p. 97]).

Now, the general philosophy of category theory suggests that there should be an appropriate notion of maps between functors, so that functors along with these maps can themselves form a category. These are precisely natural transformations $\alpha : F \Rightarrow G$, for functors $F, G : \mathcal{A} \to \mathcal{B}$. A natural transformation α consists of a family of maps $\alpha_A : F(A) \to G(A)$ in $\mathcal{B}$ for $A \in \mathsf{Ob}(\mathcal{A})$ such that for any $f : A \to A'$ in $\mathcal{A}$, we have the square

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_{A'}} \\ G(A) & \xrightarrow[G(f)]{} & G(A') \end{array}$$

which commutes in the sense that $\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A$. Natural transformations are essential for comparing functors, and will be extremely important in our construction of polynomial approximations to a functor.

Given a natural transformation $\alpha : F \Rightarrow G$ between functors $F, G : \mathcal{A} \to \mathcal{B}$, as well as functors $H : \mathcal{D} \to \mathcal{A}$ and $K : \mathcal{B} \to \mathcal{C}$, we can perform an operation called whiskering [Rie17, p. 47] to create a natural transformation $K\alpha_H : KFH \Rightarrow KGH$ given on an object $D \in \mathsf{Ob}(\mathcal{D})$ by

$$(K\alpha_H)_D := K(\alpha_{H(D)})$$

Whiskering will also be very important for constructing certain resolutions that go into our definition of polynomial functors.

Hereafter, for categories $\mathcal{A}$ and $\mathcal{B}$ we will let $\mathrm{Fun}(\mathcal{B}, \mathcal{A})$ denote the category with objects functors and maps natural transformations between functors. These will be our primary categories of study, where $\mathcal{A}$ will either be R-Mod for some ring R, or the category of chain complexes valued in R-Mod. Moving forward we will fix the notation R for a ring.

2.2 Homotopies: From Spaces to Chain Complexes

A common notion of equivalence for topological spaces seen in introductory classes is homeomorphism. However, algebraic topologists are often interested in weaker notions of equivalence. The central motivating example for this thesis is the notion of two spaces being homotopic (see e.g. [Hat02, p. 3]).

Definition 2.3 Two continuous functions $f, g : X \to Y$ between topological spaces are said to be homotopic if there exists a continuous deformation $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. In this case we write $f \simeq g$.
Two spaces X and Y are said to be homotopy equivalent, denoted $X \simeq Y$, if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

Note that the main difference between spaces being homotopy equivalent versus homeomorphic is that the composition of maps is only required to be homotopic to the identities, rather than equal.

In order to deal with weak notions of equivalences, like homotopies in topological spaces, in our setting, we need to replace our module categories by appropriate homotopical analogues where we can introduce properties up to weak equivalence. A concrete approach that can be taken, and which will be the focus of this thesis, is the introduction of chain complexes on a module category [DS95].

A chain complex $A_\bullet$ in R-Mod consists of a sequence of R-modules $A_n, n \in \mathbb{Z}$, together with module maps forming a chain

$$\cdots \to A_{n+2} \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} A_{n-2} \to \cdots \tag{1}$$

such that for any $n \in \mathbb{Z}$ $\partial_n \circ \partial_{n+1} = 0$ is the zero map, or equivalently $\mathrm{Im}(\partial_{n+1}) \subseteq \ker(\partial_n)$. Chain complexes originally appeared in abstract algebra, but since have found tremendous use in algebraic topology. In algebraic topology the degree to which $\mathrm{Im}(\partial_{n+1})$ fails to equal $\ker(\partial_n)$ is used to encode topological information about the space being considered, such as in detecting holes in the space.

To define a category of chain complexes we need maps. A map of chain complexes $f_\bullet : A_\bullet \to B_\bullet$ is a collection of maps $f_n : A_n \to B_n, n \in \mathbb{Z}$, which commute with boundaries in the sense that $\partial_n^B \circ f_n = f_{n-1} \circ \partial_n^A$. In other words, we can think of a chain map as a type of natural transformation. Together this data defines a category $\mathsf{Ch}(R\text{-}\mathsf{Mod})$ of chain complexes valued in R-Mod with chain maps between chain complexes as arrows.

We have a natural way of encoding the data in R-Mod in our new category $\mathsf{Ch}(R\text{-}\mathsf{Mod})$. This is done using a functor which sends an R-module A to the chain complex with A in degree zero and 0's elsewhere. This encoding is the first sign that $\mathsf{Ch}(R\text{-}\mathsf{Mod})$ is a good candidate for enriching R-Mod to a category in which weaker equivalences can be considered. For technical reasons associated with certain finiteness conditions we will be restricting our study to the sub-category $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ of chain complexes with non-zero entries only in degrees ≥ 0. In this setting we denote the described functor by $\deg_0^{R\text{-}\mathsf{Mod}} : R\text{-}\mathsf{Mod} \hookrightarrow \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$, and refer to it as the degree zero inclusion functor for R-Mod in $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$.

Now that we understand how to embed R-Mod into $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$, we need an appropriate notion of weak equivalence on the objects in the category $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ of chain complexes, analogous to that of homotopy equivalence for topological spaces. In this paper we will work with chain homotopy equivalences. For this and future definitions in homological algebra we will primarily refer to the text [Wei94].

Definition 2.4 [Wei94, Defn. 1.4.4] A chain homotopy between chain maps $f, g : A_\bullet \to B_\bullet$ is a collection of maps $s_n : A_n \to B_{n+1}, n \in \mathbb{Z}$, such that

$$f_n - g_n = \partial^B_{n+1} \circ s_n + s_{n-1} \circ \partial^A_n$$

In this case we write $f \simeq g$. Two chain complexes $A_\bullet$ and $B_\bullet$ are said to be chain homotopy equivalent if we have chain maps $f : A_\bullet \to B_\bullet$ and $g : B_\bullet \to A_\bullet$ such that $g \circ f \simeq 1_{A_\bullet}$ and $f \circ g \simeq 1_{B_\bullet}$.

Although this definition of chain homotopy may appear to be esoteric at first glance, after suitable reformulation it is equivalent to that of topological homotopies.

For the work in [JM04], Johnson and McCarthy consider a slightly weaker notion of equivalence than that given by chain homotopies. Explicitly, Johnson and McCarthy consider quasi-isomorphisms of chain complexes. For the purpose of this thesis the technical definition of quasi-isomorphisms is unnecessary as we will only use the well-known result in the literature that chain homotopy equivalences are necessarily quasi-isomorphisms [Wei94, Lem 1.4.5]. On the other hand, there are chain complexes which are quasi-isomorphic but not chain homotopy equivalent, so quasi-isomorphisms are strictly weaker than chain homotopy equivalences.

2.3 Mapping Cones

For our comparison with the construction in [JM04] an important tool is the mapping cone, which has origins in topology. The mapping cone can be constructed directly without reference to other concepts, but for our purposes it is best to first introduce the totalization functor which will prove essential in other parts of our construction.

The definition of the total complex functor relies on the observation that we can form chain complexes in the category of chain complexes $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ itself to obtain a category of chain complexes of chain complexes, $\mathsf{Ch}_{\geq 0}(\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod}))$. The objects of this category are called bicomplexes, $A_{\bullet,\bullet}$, and pictorially they can be represented as follows

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & A_{2,2} & \xrightarrow{\partial^h_{2,2}} & A_{1,2} & \xrightarrow{\partial^h_{1,2}} & A_{0,2} \\
 & & \downarrow{\scriptstyle \partial^v_{2,2}} & & \downarrow{\scriptstyle \partial^v_{1,2}} & & \downarrow{\scriptstyle \partial^v_{0,2}} \\
\cdots & \longrightarrow & A_{2,1} & \xrightarrow{\partial^h_{2,1}} & A_{1,1} & \xrightarrow{\partial^h_{1,1}} & A_{0,1} \\
 & & \downarrow{\scriptstyle \partial^v_{2,1}} & & \downarrow{\scriptstyle \partial^v_{1,1}} & & \downarrow{\scriptstyle \partial^v_{0,1}} \\
\cdots & \longrightarrow & A_{2,0} & \xrightarrow[\partial^h_{2,0}]{} & A_{1,0} & \xrightarrow[\partial^h_{1,0}]{} & A_{0,0}
\end{array}$$

where for each $n, m \in \mathbb{Z}$, $\partial^h_{n,m} : A_{n,m} \to A_{n-1,m}$ and $\partial^v_{n,m} : A_{n,m} \to A_{n,m-1}$ denote the horizontal and vertical boundary maps, respectively. Each row and each column of a bicomplex is a chain complex in its own right, so $\partial^h_{n,m} \circ \partial^h_{n+1,m} = 0$ and $\partial^v_{n,m} \circ \partial^v_{n,m+1} = 0$ for all $n, m \in \mathbb{Z}$. The totalization is then a functor which collapses a bicomplex of this form into a single chain complex. In particular, it provides a chain complex which encodes the vertical and horizontal chain complexes in the bicomplex simultaneously.

Definition 2.5 [Wei94, Sec. 1.2.6] We define a functor $\mathsf{Tot} : \mathsf{Ch}_{\geq 0}(\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})) \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ by sending $(A_{\bullet,\bullet}, \partial_{\bullet,\bullet}) \in \mathsf{Ch}_{\geq 0}(\mathsf{Ch}_{\geq 0}(\mathcal{A}))$ to $(\mathsf{Tot}(A_{\bullet,\bullet})_\bullet, \partial_\bullet^{\mathsf{Tot}})$ with

$$\mathsf{Tot}(A_{\bullet,\bullet})_n = A_{0,n} \oplus A_{1,n-1} \oplus \cdots \oplus A_{n,0}$$

and

$$\partial_n^{\mathsf{Tot}} : A_{0,n} \oplus A_{1,n-1} \oplus \cdots \oplus A_{n,0} \to A_{0,n-1} \oplus A_{1,n-2} \oplus \cdots \oplus A_{n-1,0},$$
$$(a_0, a_1, ..., a_n) \mapsto (\partial^h_{1,n-1}(a_1) + \partial^v_{0,n}(a_0), ..., \partial^h_{n,0}(a_n) + (-1)^n \partial^v_{n-1,1}(a_{n-1}))$$

where the alternating signs on the vertical boundary maps appear to ensure that $\partial_n^{\mathsf{Tot}}$ satisfies the boundary condition [Wei94, Sec. 1.2.5-6].
Given a map $f_{\bullet,\bullet} : A_{\bullet,\bullet} \to B_{\bullet,\bullet}$, $\mathsf{Tot}(f_{\bullet,\bullet})$ acts componentwise so

$$\mathsf{Tot}(f_{\bullet,\bullet})_n(a_0, ..., a_n) := (f_{0,n}(a_0), f_{1,n-1}(a_1), ..., f_{n,0}(a_n)),\ \forall n \geq 0, (a_0, ..., a_n) \in \mathsf{Tot}(A_{\bullet,\bullet})_n$$

Using this construction we can now define the mapping cone for a chain map, $f_\bullet : A_\bullet \to B_\bullet$. This definition relies on the observation that we can use a chain map to build a bicomplex where the vertical boundaries are the chain map, $\partial^v_{n,1} := f_n$, or zero maps, and the horizontal boundaries are given by the boundaries of $A_\bullet$, the boundaries of $B_\bullet$, or zero maps, depending on the row being considered.

Definition 2.6 [Wei94, Sec. 1.5.1] The mapping cone for a chain map $f_\bullet : A_\bullet \to B_\bullet$, denoted $\mathrm{cone}(f_\bullet)$, is given by the totalization of the bicomplex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_2 & \xrightarrow{\partial_2^A} & A_1 & \xrightarrow{\partial_1^A} & A_0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ \cdots & \longrightarrow & B_2 & \xrightarrow[\partial_2^B]{} & B_1 & \xrightarrow[\partial_1^B]{} & B_0 \end{array}$$

with $B_\bullet$ located in the 0th row, $A_\bullet$ located in the 1st row, and all other rows being 0s.

The boundary maps for the mapping cone can be described by simple 2×2 matrices

$$A_{n-1} \oplus B_n \xrightarrow{\begin{pmatrix} \partial^A_{n-1} & 0 \\ (-1)^{n-1} f_{n-1} & \partial^B_n \end{pmatrix}} A_{n-2} \oplus B_{n-1},$$
$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} \partial^A_{n-1}(a) \\ (-1)^{n-1} f_{n-1}(a) + \partial^B_n(b) \end{pmatrix}$$

when we notate the elements of the direct sums $A_{n-1} \oplus B_n$ and $A_{n-2} \oplus B_{n-1}$ as column vectors. This matrix notation will be incredibly convenient for the comparison with [JM04] as the composition of maps in this notation is exactly given by matrix multiplication.

3 Universal Polynomial Approximations

3.1 Cross-Effects

Our primary goal in this work is to construct universal approximation functors valued in chain complexes. Up to a suitable notion of homotopy of functors, made precise below, we will also show that our construction is equivalent to that in [JM04]. However, this raises an important question: what properties should our approximate functors satisfy so that they are easier to manipulate than our original abstract functors? The primary answer to this question comes from preserving the structure of spaces built up from smaller spaces.

The category of topological spaces with base points, Top_*, has two properties which are important for this work. First, since maps in Top_* are continuous maps that preserve base points, every topological space X with base point x_0 has a unique map from the singleton space $\{*\}$, sending $*$ to x_0. From this observation the singleton space $\{*\}$ has the important property that every object in Top_* has a unique map both into and out of it. In addition to this characteristic, Top_* has a construction which produces a space $X \vee Y$ from spaces X and Y by identifying the specified base points in X and Y. An invariant $F : \mathsf{Top}_* \to \mathcal{A}$ which sends $X \vee Y$ to $F(X) \oplus F(Y)$ is then valuable as it encodes this decomposition of the space (recall $\mathcal{A}$ denotes either R-Mod or $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$). This property can be considered to be a form of additivity for our functorial invariants.

In general, we will consider functors $F : \mathcal{B} \to \mathcal{A}$ where $\mathcal{B}$ is a category with properties similar to that of Top_*. In particular, we will ask that $\mathcal{B}$ has a base point object, $*$, which has unique maps into and out of all other objects. We will also ask that any two objects B, B' in $\mathcal{B}$ can be joined to form an object $B \vee B'$ with natural inclusions $B \to B \vee B' \leftarrow B'$. For example, if $\mathcal{B}$ is the category of R-modules (respectively, chain complexes of R-modules) then it has a base point object given by the 0 module (respectively, the chain of 0 modules) and a join operation given by direct sums, $\oplus$. We will fix such a category $\mathcal{B}$ hereafter, where the reader can replace $\mathcal{B}$ by Top_* or $\mathcal{A}$ throughout.

In this context F is said to be *additive* if it preserves the base point, $F(*) \cong 0$, and for any pair of objects B and B' in $\mathcal{B}$, the map formed out of the inclusions for $B \vee B'$,

$$F(B) \oplus F(B') \to F(B \vee B')$$

is an isomorphism [Mac71, p. 197]. When $\mathcal{B} = S$-Mod or $\mathsf{Ch}_{\geq 0}(S\text{-}\mathsf{Mod})$ for a ring S, $F : \mathcal{B} \to \mathcal{A}$ being additive is equivalent to $F_{B,B'} : \mathcal{B}(B, B') \to \mathcal{A}(F(B), F(B'))$ being a homomorphism of abelian groups for any objects $B, B' \in \mathsf{Ob}(\mathcal{B})$ [Mac71, p. 197]. This perspective on additivity of functors will be important for our later discussion because having an additive functor $F : \mathcal{B} \to \mathcal{A}$ allows us to extend to a functor $F : \mathsf{Ch}_{\geq 0}(\mathcal{B}) \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$ given by acting pointwise on complexes:

$$(\cdots \to B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0) \longmapsto (\cdots \to F(B_2) \xrightarrow{F(\partial_2)} F(B_1) \xrightarrow{F(\partial_1)} F(B_0))$$

Additionally, with this characterization of additivity, F sends chain homotopies to chain homotopies since the chain homotopy condition in Definition 2.4 is defined in terms of sums and composites of maps, which F preserves.[1]

In the case of functions $f : A \to B$ of abelian groups, if $f(0) = 0$, f being additive, or in other words a group homomorphism, is equivalent to the condition

$$f(x + y) - f(x) - f(y) = 0, \ \forall x, y \in A$$

Historically, the term on the left is called the second **cross-effect** of f, and is denoted by $\mathrm{cr}_2(f)$. Similarly, there are higher cross-effects which measure the failure of f to be polynomial of a specific degree. Eilenberg and MacLane generalized this definition to that

[1] See also the discussion in [Wei94, p. 391].

of functors between abelian categories in 1954 in order to measure the defect of a functor from being polynomial-like [EM54]. For our current work we will utilize the definition given in [JM04].

Definition 3.1 [JM04, Defn. 1.1] Given a functor $F : \mathcal{B} \to \mathcal{A}$, for $n \geq 1$ we define its nth cross-effect, $\mathrm{cr}_n(F) : \mathcal{B}^n \to \mathcal{A}$, inductively using the following implicit formulas:

$$\mathrm{cr}_1(F)(X) \oplus F(0) \cong F(X), \quad \forall X \in \mathsf{Ob}(\mathcal{B})$$

and if $\mathrm{cr}_{n-1}(F)$ for $n-1 \geq 1$ is defined, then $\mathrm{cr}_n(F)$ is defined implicitly by

$$\begin{aligned} \mathrm{cr}_n(F)(X_1, ..., X_n) \oplus \mathrm{cr}_{n-1}(F)(X_1, X_3, ..., X_n) \oplus \mathrm{cr}_{n-1}(F)(X_2, X_3, ..., X_n) \\ \cong \mathrm{cr}_{n-1}(F)(X_1 \vee X_2, X_3, ..., X_n), \quad \forall X_i \in \mathsf{Ob}(\mathcal{B}), 1 \leq i \leq n \end{aligned}$$

An explicit construction of cr_n can be performed inductively by defining it to be a kernel of the projection of $\mathrm{cr}_{n-1}(F)(X_1 \vee X_2, X_3, ..., X_n)$ onto the direct sum of $\mathrm{cr}_{n-1}(F)(X_1, X_3, ..., X_n)$ with $\mathrm{cr}_{n-1}(F)(X_2, X_3, ..., X_n)$. In this perspective the action of $\mathrm{cr}_n(F)$ on arrows is given by restricting maps to maps between kernels. With this data cr_n becomes a functor between functor categories

$$\mathrm{cr}_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$$

where $\mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$ denotes the category of n-variable functors that send any n-tuple containing the base point to zero. Further, from our inductive definition, cr_n is in fact an additive functor in its own right, which is to say $\mathrm{cr}_n(F \oplus G) \cong \mathrm{cr}_n(F) \oplus \mathrm{cr}_n(G)$ for $F, G : \mathcal{B} \to \mathcal{A}$ [BJO+18, Prop. 2.10].

We can now use the cross-effect to define the degree of a functor in analogy with the degree of polynomial functions. Since we will want to be able to work with weak notions of equivalence, hereafter we will consider functors $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$, where still $\mathcal{A}$ is either R-Mod or $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$. We are still able to consider all functors $G : \mathcal{B} \to \mathcal{A}$ by post-composing them with the degree zero functor $\deg_0^{\mathcal{A}} : \mathcal{A} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$, which can be vizualised as embedding G in a chain with G at the 0th position and 0's elsewhere.

A functor is said to be degree n if, as in the case of real functions, its $n+1$st cross-effect is zero, though as with our general philosophy we weaken the requirement of equality to that of equivalence. In [BJO+18, Defn. 4.2] pointwise chain homotopy equivalences are used to define when a functor is degree n, but in this work we instead use the notion of natural chain homotopy equivalences.

Definition 3.2 A functor $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$ is said to be of degree n if $\mathrm{cr}_{n+1}(F)$ is naturally chain homotopy equivalent to the zero functor. In this case we say that $\mathrm{cr}_{n+1}(F)$ is **contractible**.

In Definition 3.2 natural chain homotopy equivalence is a combination of the mapping structure for functors, natural transformations, and the equivalence structure for chain complexes, chain homotopy equivalences. Explicitly, a natural chain homotopy equivalence is exactly a chain homotopy equivalence under the isomorphism of categories in Proposition 3.3. Note that just as in a general category, an isomorphism of categories is a functor with an inverse. The pointwise chain homotopy equivalences in [BJO+18], on the other hand, are strictly weaker than natural chain homotopy equivalences as they do not require that the homotopy maps from Definition 2.4 live in the category of functors and natural transformations.

Proposition 3.3 We have an isomorphism of categories

$$\mathsf{Ch}_{\geq 0}(\mathrm{Fun}(\mathcal{B},\mathcal{A})) \cong \mathrm{Fun}(\mathcal{B},\mathsf{Ch}_{\geq 0}(\mathcal{A})) \tag{2}$$

given by a relabeling of terms.

Proof. We define a functor $\gamma : \mathsf{Ch}_{\geq 0}(\mathrm{Fun}(\mathcal{B},\mathcal{A})) \to \mathrm{Fun}(\mathcal{B},\mathsf{Ch}_{\geq 0}(\mathcal{A}))$ given on $F_\bullet$ in $\mathsf{Ch}_{\geq 0}(\mathrm{Fun}(\mathcal{B},\mathcal{A}))$ by setting $\gamma(F_\bullet)$ to be the functor defined on B and $f : B \to B'$ in $\mathcal{B}$ by

$$\gamma(F_\bullet)(B)_n := F_n(B) \quad \text{and} \quad \gamma(F_\bullet)(f)_n := F_n(f),\ \forall n \geq 0$$

where the differentials of $\gamma(F_\bullet)(B)$ are the natural transformation differentials $\partial_n : F_n \Rightarrow F_{n-1}$ evaluated at B. Since the differentials ∂_n are natural and each F_n is a functor, $\gamma(F_\bullet)$ is a functor. Next, for $\alpha_\bullet : F_\bullet \to G_\bullet$ we define $\gamma(\alpha_\bullet) : \gamma(F_\bullet) \Rightarrow \gamma(G_\bullet)$ to have components

$$(\gamma(\alpha_\bullet)_B)_n := (\alpha_n)_B,\ \ \forall B \in \mathsf{Ob}(\mathcal{B}),\ \forall n \geq 0$$

For each $B \in \mathsf{Ob}(\mathcal{B})$, $\gamma(\alpha_\bullet)_B$ defines a chain map $\gamma(F_\bullet)(B) \to \gamma(G_\bullet)(B)$ since $\alpha_\bullet$ is a chain map of natural transformations, so all squares with differentials commute. Further, $\gamma(\alpha_\bullet)$ is natural since if $f : B \to B'$ is a map in $\mathcal{B}$, then at each $n \geq 0$

$$\gamma(G_\bullet)(f)_n \circ (\gamma(\alpha_\bullet)_B)_n = G_n(f) \circ (\alpha_n)_B = (\alpha_n)_{B'} \circ F_n(f) = (\gamma(\alpha_\bullet)_{B'})_n \circ \gamma(F_\bullet)(f)_n$$

using naturality of α_n. Since this definition is in terms of the components of $\alpha_\bullet$ it is inherently functorial as composition of natural transformations is defined componentwise.

Next we must witness an inverse functor $\rho : \mathrm{Fun}(\mathcal{B},\mathsf{Ch}_{\geq 0}(\mathcal{A})) \to \mathsf{Ch}_{\geq 0}(\mathrm{Fun}(\mathcal{B},\mathcal{A}))$. For $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$ we set $\rho(F)$ to have nth component functor $(-)_n \circ F$ where $(-)_n : \mathsf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}$ is the functor sending a chain complex to its nth term. The differential $\partial_n^F : (-)_n \circ F \Rightarrow (-)_{n-1} \circ F$ is given at $B \in \mathsf{Ob}(\mathcal{B})$ by the nth differential in $F(B)$, $\partial_n^B : F(B)_n \to F(B)_{n-1}$. Naturality of the differential equates to the commutivity of the square

$$\begin{array}{ccc} F(B)_n & \xrightarrow{\partial_n(B)} & F(B)_{n-1} \\ {\scriptstyle F(f)_n}\downarrow & & \downarrow{\scriptstyle F(f)_{n-1}} \\ F(B')_n & \xrightarrow[\partial_n(B')]{} & F(B')_{n-1} \end{array}$$

for any $f : B \to B'$, which follows since $F(f)$ is a chain map.

Finally, we must define ρ on maps. If $\alpha : F \Rightarrow G$ is a natural transformation between two functors $F, G : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$, we define $\rho(\alpha)$ by setting $\rho(\alpha)_n$, $n \geq 0$, to be the natural transformation with components $(\rho(\alpha)_n)_B := (\alpha_B)_n$ for $B \in \mathsf{Ob}(\mathcal{B})$. For each $n \geq 0$, $\rho(\alpha)_n$ is natural since for any $f : B \to B'$ in $\mathcal{B}$, we can compute

$$\rho(G)_n(f) \circ (\rho(\alpha)_n)_B = G(f)_n \circ (\alpha_B)_n = (\alpha_{B'})_n \circ F(f)_n = (\rho(\alpha)_n)_{B'} \circ \rho(F)_n(f)$$

using naturality of α. Additionally, $\rho(\alpha)$ is a map of chain complexes since for any $n \geq 0$ and any $B \in \mathsf{Ob}(\mathcal{B})$,

$$(\partial_n^G)_B \circ (\rho(\alpha)_n)_B = \partial_n^{G(B)} \circ (\alpha_B)_n = (\alpha_B)_{n-1} \circ \partial_n^{F(B)} = (\rho(\alpha)_{n-1})_B \circ (\partial_n^F)_B$$

using the fact that α_B is a chain map for every $B \in \mathsf{Ob}(\mathcal{B})$.

Once again, since $\rho(\alpha)$ is defined in terms of the components of α, the assignment ρ is inherently functorial. Further, γ and ρ are exactly inverse of each other as they correspond to simply swapping the element and natural number indices. ■

This proof suggests that our choice of natural chain homotopy equivalences is the correct notion for studying polynomial functors in terms of chain homotopy since it provides exactly the chain homotopies internal to a chain complex category.

With the notion of degree n functors and natural chain homotopy equivalences now defined, we can begin approximating a functor by a sequence of functors of increasing degree, in analogy with the approximation of real functions by their Taylor polynomials. In order to perform this approximation we require a definition of universal degree n functors which will act as our Taylor polynomials. In [JM04, p. 769] universal degree n functors are defined in terms of quasi-isomorphisms, which [BJO+18] expands to pointwise chain homotopy equivalences. From our work in Proposition 3.3, we now generalize the definition in [JM04, p. 769] to natural chain homotopy equivalences.

Definition 3.4 For a functor $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$, a degree n functor $G : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$ together with a natural transformation $\alpha : F \Rightarrow G$ is said to be a universal degree n approximation of F if for any other such pair (H, β), the following hold:
(i) there exists a transformation $\gamma : G \Rightarrow H$ such that $\gamma \circ \alpha \simeq \beta$, and
(ii) γ is unique up to **natural** chain homotopy equivalence.

3.2 Universality

Throughout the remainder of this paper we seek to construct an explicit model for the degree n approximation to any functor F following the work in [BJO+18], and demonstrate how it agrees with the definition given in [JM04]. This process will rely on defining operations on functors, or more explicitly functors between functor categories. We have already seen such a functor in the nth cross-effect which can be realized as a functor $\mathrm{cr}_n : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$. We turn this functor into a functor with codomain $\mathrm{Fun}(\mathcal{B}, \mathcal{A})$ by post-composing with the diagonal functor

$$\Delta^* : \mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{A}), \quad \Delta^*(F)(X) := F(X, ..., X)$$

Note that as discussed previously for the cross-effects, Δ^* is additive since evaluating a multi-variable functor at the same object in all variables preserves the direct sum operation.

Together these functors define what is known as a comonad [Rie17, Defn. 5.1.6] on the category of functors for each $n \geq 1$ given by $C_n := \Delta^* \mathrm{cr}_n$.

Definition 3.5 A functor $C : \mathcal{C} \to \mathcal{C}$ is said to be a comonad on the category $\mathcal{C}$ if there exist natural transformations called the co-multiplication, $\delta : C \Rightarrow C^2$, and co-unit, $\epsilon : C \Rightarrow 1_{\mathcal{C}}$, which satisfy certain equational identities mimicking an associative law, $C\delta \circ \delta = \delta_C \circ \delta$, as well as unital laws, $C\epsilon \circ \delta = 1_C$ and $1_C = \epsilon_C \circ \delta$.

In addition to these transformations and identities, since C_n is constructed in a special way as a composition of the functors Δ^* and cr_n, it also comes equipped with a natural transformation called the unit, $\eta : 1_{\mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})} \Rightarrow \mathrm{cr}_n \Delta^*$, which will be used in showing that the nth polynomial approximation we construct is degree n.

Using this structure we can begin defining the nth polynomial approximation, as seen in [BJO+18]. First we provide a preliminary definition/lemma for comonads on categories of modules or chains of modules, $\mathcal{A}$.

Lemma 3.6 If (C, δ, ϵ) is a comonad on $\mathcal{A}$, then we have a functor $C^{\bullet+1} : \mathcal{A} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$ given on an object $A \in \mathsf{Ob}(\mathcal{A})$ by

$$\cdots \to C^3(A) \xrightarrow{\epsilon_{C^2(A)} - C\epsilon_{C(A)} + C^2\epsilon_A} C^2(A) \xrightarrow{\epsilon_{C(A)} - C\epsilon_A} C(A) \tag{3}$$

where in general the differentials are defined by alternating sums $\sum_{i=0}^{n-1}(-1)^i C^i \epsilon_{C^{n-i}(A)}$.

Proof. From [Wei94, Defn. 8.2.1] and [Wei94, Defn. 8.6.4], the functor $C^{\bullet+1}$ is exactly the composite of two well-known functors in homological algebra, the unnormalized chain complex functor and the simplicial object functor for the comonad C, and so itself is a functor. ■

Mixing the approach in [JM04] and [BJO+18] we can then define the complex $C_n^\bullet(F)$ for a functor $F : \mathcal{B} \to \mathcal{A}$ as the mapping cone of the map of chain complexes, for $B \in \mathsf{Ob}(\mathcal{B})$,

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_n^3(F)(B) & \longrightarrow & C_n^2(F)(B) & \longrightarrow & C_n(F)(B) \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \epsilon_{F(B)}} \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & F(B)
\end{array}$$

Although slightly different in form, this definition produces the same result as in [BJO+18], which is the chain complex $C_n^{\bullet+1}(F)$ augmented by $\epsilon_F : C_n(F) \to F$. For $n \geq 0$ this procedure defines a functor $C_n^\bullet : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}_{\geq 0}(\mathcal{A}))$. Defining this functor using the mapping cone allows for a clearer comparison with the definition in [JM04]. Now, as in [BJO+18, Defn. 4.2] we can define the nth polynomial approximation as a functor on functor categories.

Definition 3.7 The nth polynomial approximation for functors from $\mathcal{B}$ to $\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ is defined as the composite functor

$$P_n := (\mathsf{Tot})_* \circ C_{n+1}^\bullet : \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})) \to \mathrm{Fun}(\mathcal{B}, \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})) \tag{4}$$

Here $(\mathsf{Tot})_*$ is the functor defined by post-composition with the totalization functor, so $(\mathsf{Tot})_*(F) = \mathsf{Tot} \circ F$ for $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod}))$.

Our next main tasks are to show agreement with [JM04] and show the universality of the construction in the sense of Definition 3.4. In order to introduce the [JM04] definition we require some preliminary definitions. First, instead of using the chain complex construction given in Lemma 3.6, Johnson and McCarthy use the **normalized chain complex** construction.

Definition 3.8 [Wei94, Defn. 8.3.6] If (C, δ, ϵ) is a comonad on $\mathcal{A}$, the normalized chain complex functor $N_C^{\bullet+1} : \mathcal{A} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$ is defined on $A \in \mathsf{Ob}(\mathcal{A})$ by

$$N_C^{\bullet+1}(A)_n := \bigcap_{i=0}^{n-1} \ker(C^i \epsilon_{C^{n-i}(A)}), \ \forall n \geq 1$$

and

$$N_C^{\bullet+1}(A)_0 := C(A)$$

where boundary maps are given by $C^n\epsilon_A$ for $n \geq 1$.

From the construction of the normalized chain complex, Johnson and McCarthy define their nth polynomial approximation for a functor $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ by first taking the mapping cone of the chain map

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \ker(\epsilon_{C_n^2(F)(B)}) \cap \ker(C_n \epsilon_{C_n(F)(B)}) & \longrightarrow & \ker(\epsilon_{C_n(F)(B)}) & \longrightarrow & C_n(F)(B) \\ & & \downarrow & & \downarrow & & \downarrow \epsilon_{F(B)} \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & F(B) \end{array}$$

and then totalizing. In order to show that this construction is equivalent to the one in Definition 3.7, we first prove a lemma on mapping cones and chain homotopies.

Lemma 3.9 Let $f_\bullet : A_\bullet \to B_\bullet$ be a chain map in $\mathsf{Ch}_{\geq 0}(\mathcal{A})$. Then if $g_\bullet, h_\bullet : D_\bullet \to A_\bullet$ are chain homotopic via $s_n : D_n \to A_{n+1}, n \geq 0$, with $g_\bullet \circ f_\bullet = h_\bullet \circ f_\bullet$ and $f_n \circ s_{n-1} = 0$ for $n \geq 0$, then the induced maps on the mapping cones are chain homotopic

Proof. Let $k_\bullet = g_\bullet \circ f_\bullet = h_\bullet \circ f_\bullet$. Then $g_\bullet$ and $h_\bullet$ induce chain maps $\text{cone}(k_\bullet) \to \text{cone}(f_\bullet)$ given by the diagonal matrices $\text{diag}(h_{n-1}, 1_{B_n})$ and $\text{diag}(g_{n-1}, 1_{B_n})$, for $n \geq 0$. These are chain maps since

$$\begin{pmatrix} \partial^A_{n-1} & 0 \\ (-1)^{n-1} f_{n-1} & \partial^B_n \end{pmatrix} \begin{pmatrix} h_{n-1} & 0 \\ 0 & 1_{B_n} \end{pmatrix} = \begin{pmatrix} \partial^A_{n-1} h_{n-1} & 0 \\ (-1)^{n-1} f_{n-1} h_{n-1} & \partial^B_n \end{pmatrix} = \begin{pmatrix} h_{n-1} & 0 \\ 0 & 1_{B_n} \end{pmatrix} \begin{pmatrix} \partial^C_{n-1} & 0 \\ (-1)^{n-1} k_{n-1} & \partial^B_n \end{pmatrix}$$

and

$$\begin{pmatrix} \partial^A_{n-1} & 0 \\ (-1)^{n-1} f_{n-1} & \partial^B_n \end{pmatrix} \begin{pmatrix} g_{n-1} & 0 \\ 0 & 1_{B_n} \end{pmatrix} = \begin{pmatrix} \partial^A_{n-1} g_{n-1} & 0 \\ (-1)^{n-1} f_{n-1} g_{n-1} & \partial^B_n \end{pmatrix} = \begin{pmatrix} g_{n-1} & 0 \\ 0 & 1_{B_n} \end{pmatrix} \begin{pmatrix} \partial^C_{n-1} & 0 \\ (-1)^{n-1} k_{n-1} & \partial^B_n \end{pmatrix}$$

using the fact that h_{n-1} and g_{n-1} are chain maps and the definition of $k_\bullet$. Then the matrix $\begin{pmatrix} s_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$ for $n \geq 0$ defines our desired chain homotopy since

$$\begin{aligned} \begin{pmatrix} g_{n-1} & 0 \\ 0 & 1_{B_n} \end{pmatrix} - \begin{pmatrix} h_{n-1} & 0 \\ 0 & 1_{B_n} \end{pmatrix} &= \begin{pmatrix} g_{n-1} - h_{n-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \partial^A_n \circ s_{n-1} + s_{n-2} \circ \partial^C_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \partial^A_n & 0 \\ (-1)^n f_n & \partial^B_{n+1} \end{pmatrix} \begin{pmatrix} s_{n-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} s_{n-2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial^C_{n-1} & 0 \\ (-1)^{n-1} k_{n-1} & \partial^B_n \end{pmatrix} \end{aligned}$$

where in the last equality we use the fact that $f_n \circ s_{n-1} = 0$ for $n \geq 0$. ■

Since Tot preserves chain homotopies [Wei94, p. 147], Lemma 3.9 allows us to prove the equivalence of the constructions in [JM04] and [BJO+18] up to natural chain homotopy.

Proposition 3.10 For all $n \geq 0$, the functors $C_n^\bullet$ and $N_{C_n}^\bullet$ are naturally chain homotopy equivalent.

Proof. A well-known result in simplicial homotopy theory says that the construction $C_n^{\bullet+1}$ and the construction $N_{C_n}^{\bullet+1}$ are naturally chain homotopy equivalent via the inclusion of $N_{C_n}^{\bullet+1}$ into $C_n^{\bullet+1}$ [GJ09, Thm 2.5]. Therefore, in order to show that our definition, $C_n^\bullet$, coincides with the definition in [JM04], $N_{C_n}^\bullet$, up to natural chain homotopy equivalence, we need only show that the mapping cone construction preserves our chain homotopies. Since the vertical maps for the mapping cone are all zeros except in the zeroth degree, in which they agree, and the homotopy in [GJ09, Thm 2.5] is given by the identity in degree 0, Lemma 3.9 can be applied as its hypotheses will hold for both homotopies in the equivalence. ■

Now that we know our construction coincides with that in [JM04] and [BJO$^+$18] up to natural chain homotopy, it remains to show that it gives universal degree n approximations, as defined in Definition 3.4. Towards proving this claim we first need maps $p_{n,F} : F \Rightarrow P_n(F)$ for $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ which are natural in F. This is done using the following map of chain complexes.

$$\begin{array}{ccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow & F \\
 & \downarrow & & \downarrow & & \downarrow & \downarrow 1_F \\
\cdots \longrightarrow & C^3_{n+1}(F) & \xrightarrow[\epsilon_{C^2_{n+1}(F)} - C_{n+1}\epsilon_{C_{n+1}(F)} + C^2_{n+1}\epsilon_F]{} & C^2_{n+1}(F) & \xrightarrow[\epsilon_{C_{n+1}(F)} - C_{n+1}\epsilon_F]{} & C_{n+1}(F) \xrightarrow[\epsilon_F]{} & F
\end{array}$$

The map of complexes as defined is natural as all components are either zeros or identities. Additionally, the resulting $p_{n,F}$ is obtained from this natural transformation by whiskering with the totalization functor, and so is also natural. With this map in hand the universality claim is the content of the following theorem which generalizes Proposition 4.5 in [BJO$^+$18].

Theorem 3.11 For $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$,

(i) The functor $P_n(F) : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ is degree n.

(ii) If F is degree n, then the map $p_{n,F} : F \Rightarrow P_n(F)$ is a **natural** chain homotopy equivalence.

(iii) The pair $(P_n(F), p_{n,F} : F \Rightarrow P_n(F))$ is universal up to **natural** chain homotopy equivalence with respect to degree n functors with maps from F.

Proof. Throughout let $R = \mathrm{cr}_{n+1}$ and let $L = \Delta^*$, so $C_{n+1} = LR$. Additionally, let $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ be a functor. We proceed with the proof in parts.

(i) To show that $P_n(F)$ is degree n we must show that the composite $R \circ (\mathsf{Tot})_* \circ C^\bullet_{n+1}$ is contractible. However, since R is additive and Tot is defined in terms of direct sums, we have a natural isomorphism

$$R \circ (\mathsf{Tot})_* \circ C^\bullet_{n+1}(F) \cong (\mathsf{Tot})_* \circ R \circ C^\bullet_{n+1}(F)$$

Additionally, as Tot preserves chain homotopies [Wei94, p. 147] it is sufficient to show that $R \circ C^\bullet_{n+1}(F)$ is contractible. However, as shown generally for a comonad in Lemma 2.9 of [BJO$^+$18], $R \circ C^\bullet_{n+1}$ has a natural chain homotopy between the identity and 0 maps, $s_k : RC^k_{n+1} \Rightarrow RC^{k+1}_{n+1}$, given by $\eta_{RC^k_{n+1}}$, so $R \circ C^\bullet_{n+1}$ is contractible and $P_n(F)$ is degree n.

(ii) Suppose $F : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$ is of degree n, so that $R(F)$, and hence $C_{n+1}(F) = LR(F)$ is naturally chain homotopy equivalent to the zero functor. Since C_{n+1} is additive it preserves chain homotopies, so $C^k_{n+1}(F)$ is chain contractible for all $k \geq 1$. Then $C^\bullet_{n+1}(F)$ is a bicomplex where every row except the zeroth row is contractible. As is shown in Corollary A.7 of [BJO$^+$18], such a bicomplex contracts to its zeroth row after totalization. But, since $p_{n,F} : F \Rightarrow P_n(F)$ was exactly defined to be the inclusion of the zeroth row of the bicomplex for $P_n(F)$ followed by a totalization, this result implies that $p_{n,F} : F \Rightarrow P_n(F)$ is a chain homotopy equivalence, as desired.

(iii) To prove the final claim we fix a degree n functor $G : \mathcal{B} \to \mathsf{Ch}_{\geq 0}(R\text{-}\mathsf{Mod})$. Next, let $\tau : F \Rightarrow G$ be a natural transformation. Then by naturality of p_n we have the following commuting square, where in diagrams we will denote natural transformations by $\to$

$$\begin{array}{ccc} F & \xrightarrow{\tau} & G \\ {\scriptstyle p_{n,F}}\downarrow & & \downarrow{\scriptstyle p_{n,G}} \\ P_n(F) & \xrightarrow[P_n(\tau)]{} & P_n(G) \end{array}$$

Let $s_{n,G} : P_n(G) \Rightarrow G$ be the natural homotopy inverse of $p_{n,G}$ which exists by property (ii) and the definition of chain homotopy equivalence. Setting $\tau^{\#} = s_{n,G} \circ P_n(\tau)$ we obtain

$$\tau^{\#} \circ p_{n,F} = s_{n,G} \circ P_n(\tau) \circ p_{n,F} = s_{n,G} \circ p_{n,G} \circ \tau \simeq \tau$$

since $s_{n,G} \circ p_{n,G} \simeq 1_G$, so τ factors through $p_{n,F}$ up to a natural chain homotopy equivalence.

To show uniqueness for the universal property we can show that any $\sigma : P_n(F) \Rightarrow G$ such that $\sigma \circ p_{n,F} \simeq \tau$ can be written, up to natural chain homotopy equivalence, in terms of τ and maps associated with F and G. Using naturality we have a commuting rectangle

$$\begin{array}{ccccc} F & \xrightarrow{p_{n,F}} & P_n(F) & \xrightarrow{\sigma} & G \\ {\scriptstyle p_{n,F}}\downarrow & & {\scriptstyle s_{n,P_n(F)}}\dashuparrow\ \downarrow{\scriptstyle p_{n,P_n(F)}} & & {\scriptstyle s_{n,G}}\dashuparrow\ \downarrow{\scriptstyle p_{n,G}} \\ P_n(F) & \xrightarrow[P_n(p_{n,F})]{} & P_n(P_n(F)) & \xrightarrow[P_n(\sigma)]{} & P_n(G) \end{array} \tag{5}$$

where from part (ii) the maps $p_{n,P_n(F)}$ and $p_{n,G}$ are natural chain homotopy equivalences. Further, from [BJO$^+$18, Prop. 4.5] there exists a natural isomorphism $\alpha : P_n^2 \Rightarrow P_n^2$ such that $\alpha \circ P_n(p_{n,F}) = p_{n,P_n(F)}$. This equality implies that $P_n(p_{n,F})$ is also a natural chain homotopy equivalence since we can write

$$P_n(p_{n,F})s_{n,P_n(F)}\alpha = \alpha^{-1}\alpha P_n(p_{n,F})s_{n,P_n(F)}\alpha = \alpha^{-1}p_{n,P_n(F)}s_{n,P_n(F)}\alpha \simeq \alpha^{-1}\alpha = 1_{P_n^2(F)} \tag{6}$$

We can now write σ in terms of just τ, α, and maps associated with F and G:

$$\begin{aligned} \sigma &\simeq s_{n,G}p_{n,G}\sigma \\ &= s_{n,G}P_n(\sigma)p_{n,P_n(F)} && \text{(by commutivity of the square in Eq. (5))} \\ &\simeq s_{n,G}P_n(\sigma)P_n(p_{n,F})s_{n,P_n(F)}\alpha p_{n,P_n(F)} && \text{(by Eq. (6))} \\ &= s_{n,G}P_n(\sigma p_{n,F})s_{n,P_n(F)}\alpha p_{n,P_n(F)} \\ &\simeq s_{n,G}P_n(\tau)s_{n,P_n(F)}\alpha p_{n,P_n(F)} \end{aligned}$$

where in the last natural chain homotopy equivalence we use the fact that $\sigma \circ p_{n,F} \simeq \tau$ and P_n preserves chain homotopies as C_{n+1} and Tot do. Since this representation only depended on $\sigma \circ p_{n,F} \simeq \tau$, we must have $\sigma \simeq \tau^{\#}$, proving uniqueness. ■

The universality in Theorem 3.11 is stronger than that given in [BJO$^+$18] or [JM04] since in [BJO$^+$18] the homotopies are not assumed to be natural, while in [JM04] quasi-isomorphisms are used. Since every homotopy equivalence is a quasi-isomorphism, this means that Theorem 3.11 is strictly stronger than Lemma 2.11 in [JM04].

3.3 Examples

In this final section we provide two examples of the first polynomial approximation of a functor. The first functor we consider is the identity functor $1_{R\text{-Mod}} : R\text{-Mod} \to R\text{-Mod}$, which we view as a functor into chain complexes via the degree zero inclusion $\deg_0^{R\text{-Mod}} : R\text{-Mod} \to \mathsf{Ch}_{\geq 0}(R\text{-Mod})$.

Example 3.1 (Degree 0) :

Note that the chain complex $\deg_0^{R\text{-Mod}}(0)$ is the zero complex. This implies that $\mathrm{cr}_1(\deg_0^{R\text{-Mod}}) \cong \deg_0^{R\text{-Mod}}$. Then, by our inductive formula for $A, B \in \mathsf{Ob}(R\text{-Mod})$,

$$\mathrm{cr}_2(\deg_0^{R\text{-Mod}})(A,B) \oplus \deg_0^{R\text{-Mod}}(A) \oplus \deg_0^{R\text{-Mod}}(B) \cong \deg_0^{R\text{-Mod}}(A \oplus B)$$

so $\mathrm{cr}_2(\deg_0^{R\text{-Mod}})(A,B) \cong 0$ since $\deg_0^{R\text{-Mod}}(A \oplus B) = \deg_0^{R\text{-Mod}}(A) \oplus \deg_0^{R\text{-Mod}}(B)$. It follows that $P_1(\deg_0^{R\text{-Mod}})_n \cong 0$ for each $n \geq 1$, and $P_1(\deg_0^{R\text{-Mod}})_0 = 1_{R\text{-Mod}}$, so

$$P_1(\deg_0^{R\text{-Mod}})(A) := \cdots \to 0 \to 0 \to A = \deg_0^{R\text{-Mod}}(A)$$

This aligns with our expectations since $\deg_0^{R\text{-Mod}}$ should itself be a "degree 0" polynomial functor, so all of its approximations should equal it, possibly up to homotopy.

Next we can consider a more non-trivial example. If $M \in \mathsf{Ob}(R\text{-Mod})$, then we have a module $M \otimes_R M$ given by tensoring M by itself over R. This construction corresponds to the familiar operation in linear algebra, where R-linear maps out of $M \otimes_R M$ are equivalent to R-bilinear maps out of $M \times M$. Since $\otimes$ distributes over sums, this functor gives an interesting example for our polynomial approximation [Alu09, p. 506].

Example 3.2 (Tensor power) :

First, observe that $0 \otimes_R 0 \cong 0$, since bilinear maps out of 0×0 are the same as linear maps out of 0. Again this implies that $\mathrm{cr}_1(- \otimes_R -) \cong - \otimes_R -$. On the other hand, if $M, N \in \mathsf{Ob}(R\text{-Mod})$, since tensors distribute over sums, our inductive formula says

$$\begin{aligned}\mathrm{cr}_2(- \otimes_R -)(M,N) \oplus (M \otimes_R M) \oplus (N \otimes_R N) &\cong (M \oplus N) \otimes_R (M \oplus N)\\ &\cong (M \otimes_R M) \oplus (M \otimes_R N) \oplus (N \otimes_R M) \oplus (N \otimes_R N)\end{aligned}$$

so $\mathrm{cr}_2(- \otimes_R -)(M,N) \cong (M \otimes_R N) \oplus (N \otimes_R M)$ is our "cross-term", as in the name of the cross-effect.
It follows that for $k \geq 1$, $C_2^k(- \otimes_R -)(M) = (M \otimes_R M)^{\oplus 2k}$, since C_2 is additive, and we get a first polynomial approximation given by increasing sums of tensor powers:

$$P_1(- \otimes_R -)(M) = \cdots \to (M \otimes_R M)^{\oplus 4} \to (M \otimes_R M) \oplus (M \otimes_R M) \to M \otimes_R M$$

4 Conclusion

In this thesis we constructed a formula for building polynomial approximations to invariants valued in chain complexes, following an approach analogous to that of Taylor series in calculus. Throughout the construction we introduced a number of important definitions and results from the theory of homological algebra, including the mapping cone and totalization functor. Using these tools we showed that our construction agreed with previous constructions found in [JM04] and [BJO+18] up to our weak equivalence of interest, natural chain homotopy equivalence. Further, we showed that our constructed polynomial functors satisfied a universal property which extended those described in [JM04] and [BJO+18]. Finally, to illuminate our construction we computed the first degree polynomial approximations of the identity and tensor product functors, which also helped illustrate the notion of degree and the origin of the cross-effect term in our work.

This thesis and the use of natural chain homotopy equivalences is part of a larger project which involves upgrading the work in [BJO+18] to connect the constructions to those used in the preprint [BBC23]. Future work to this effect involves demonstrating that the construction given in this thesis preserves composition of invariants up to our desired notion of weak equivalence, natural chain homotopy equivalence.

References

[Alu09] Paolo Aluffi, *Algebra: Chapter 0*, Graduate studies in mathematics, American Mathematical Society, 2009.

[BBC23] Kristine Bauer, Matthew Burke, and Michael Ching, *Tangent infinity-categories and Goodwillie calculus*, 2023, arXiv:2101.07819 [math.CT].

[BJO+18] Kristine Bauer, Brenda Johnson, Christina Osborne, Emily Riehl, and Amelia Tebbe, *Directional derivatives and higher order chain rules for abelian functor calculus*, Topology and its Applications **235** (2018), 375–427.

[DS95] William G. Dwyer and Jan Spalinski, *Chapter 2 - homotopy theories and model categories*, Handbook of Algebraic Topology (I.M. JAMES, ed.), North-Holland, Amsterdam, 1995, pp. 73–126.

[EM54] Samuel Eilenberg and Saunders MacLane, *On the groups $H(\prod, n)$, II: Methods of computation*, Annals of Mathematics **60** (1954), no. 1, 49–139.

[GJ09] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Basel, 2009.

[Goo90] Thomas G. Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory*, K-Theory **4** (1990), no. 1, 1 – 27.

[Goo92] ______, *Calculus II: Analytic Functors*, K-Theory **5** (1991/1992), no. 4, 295 – 332.

[Goo03] ______, *Calculus III: Taylor Series*, Geometry & Topology **7** (2003), no. 2, 645 – 711.

[Hat02] Allen Hatcher, *Algebraic topology*, Algebraic Topology, Cambridge University Press, 2002.

[JM04] Brenda Johnson and Randy McCarthy, *Deriving calculus with cotriples*, Transactions of the American Mathematical Society **356** (2004), no. 2, 757–803 (eng).

[Mac71] Saunders MacLane, *Categories for the working mathematician*, Springer-Verlag, New York, 1971, Graduate Texts in Mathematics, Vol. 5.

[Rie17] Emily Riehl, *Category theory in context*, Aurora: Dover Modern Math Originals, Dover Publications, 2017.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1994.

[Wei99] Charles A. Weibel, *Chapter 28 - history of homological algebra*, History of Topology (I.M. James, ed.), North-Holland, Amsterdam, 1999, pp. 797–836.