Modeling Operator Algebras: The Formalization of Physical Theories

E Ea T[1]

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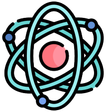
Math 617 Presentation

Roadmap



Physical Motivation/History - 5 Slides

- Formalization of Quantum Mechanics
- Generalization and abstraction

Introduction of C^*-Algebras - 3 Slides

- What is a C^*-algebra?
- How do C^*-algebras communicate?
- How can we represent it concretely?

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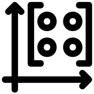

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- States: the abstraction of measurement
- Using states to construct concrete models

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- What does this tell us?
- What more can we say?

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Origin of Operator Theory

Physical Timeline

- mid-1920s: Schrödinger and Heisenberg separately construct probabilistic models of atomic phenomena
- 1930s: Stone, von Neumann, and Murray begin formalizing the theory using the budding field of operator algebras

Figure: Erwin Schrödinger and Werner Heisenberg.

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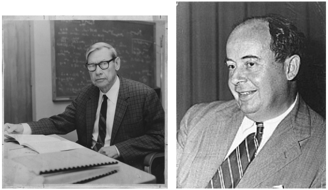

Figure: Marshall Stone and John von Neumann.

Figure: Joseph Murray

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Bounded Hilbert Space Operators

Question

How were Hilbert space operators used to formalize quantum theory?

- $\mathscr{H}$ = the state space of a physical system
- Self-adjoint $A \in \mathscr{B}(\mathscr{H})$ = observable quantities
- For $\mathcal{O} \in \mathscr{B}(\mathscr{H})$, and $\psi \in \mathscr{H}$, $\langle \mathcal{O}\psi, \psi \rangle$ = measurement of $\mathcal{O}$ for the state ψ

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Example: $\mathscr{H} = L^2(\mathbb{R})$, $\mathcal{O} = \hat{x} : \psi \mapsto x\psi$

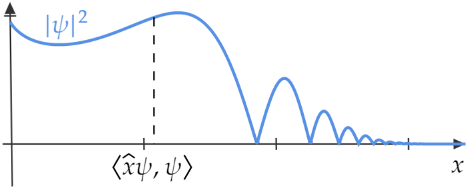

Figure: Probability distribution

Origin of Operator Theory: Abstraction

Physical Timeline

- early-1940s: Gelfand and Naimark characterize what are now known as C^*-algebras
- late-1940s: Segal demonstrated the significance of C^*-algebras to physical theory

Figure: Israel Gelfand

Figure: Mark Naimark

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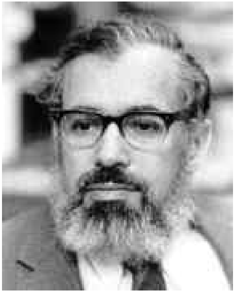

Figure: Irving Segal

What is a C^*-algebra?

Def$^{\text{n}}$: C^*-Algebra

A C^***-algebra** $\mathfrak{A}$ is a Banach space over $\mathbb{C}$ together with a multiplication $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ and an involution $^* : \mathfrak{A} \to \mathfrak{A}$ such that $\forall A, B \in \mathfrak{A}$, $\forall \alpha \in \mathbb{C}$:

C1. $||A \cdot B|| \leq ||A|| \, ||B||$

C2. $A^{**} = A$ (involutive)

C3. $(A \cdot B)^* = B^* \cdot A^*$

C4. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$ (conjugate-linearity)

C5. $||A^* \cdot A|| = ||A||^2$ (C^*-condition)

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Motivation for C5. If $A \in \mathscr{B}(\mathscr{H})$, for $\mathscr{H}$ a Hilbert space,

$$||AA^*|| = ||A||^2$$

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Morphisms of C^*-algebras

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Def$^{\text{n}}$: C^*-Algebra Homomorphism

A $*$-**homomorphism** of C^* algebras is a $\mathbb{C}$-linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ such that for all $A, B \in \mathfrak{A}$,

$$\varphi(AB) = \varphi(A)\varphi(B) \quad \text{and} \quad \varphi(A^*) = \varphi(A)^*$$

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Representation of C^*-algebras

Def^n: Representation

A **representation** of a C^*-algebra $\mathfrak{A}$ is a pair $(\varphi, \mathscr{H})$ for $\mathscr{H}$ a Hilbert space and

$$\varphi : \mathfrak{A} \to \mathscr{B}(\mathscr{H})$$

a $*$-homomorphism.

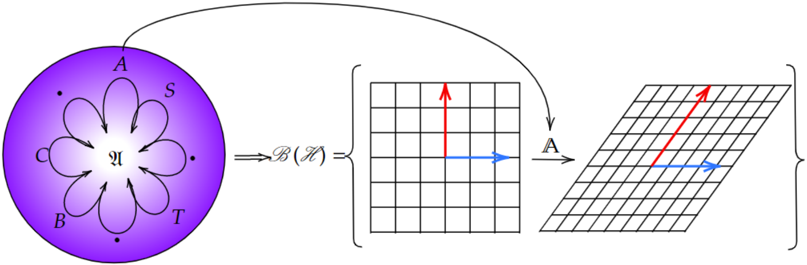

Figure: Diagram of C^*-algebra representation

Examples of Representations

Example: Sub-algebra

If $\mathscr{C} \leq \mathscr{B}(\mathscr{H})$ is a closed linear subspace space that is also algebraically closed under $\cdot$ and *, then the inclusion

$$\iota_{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathscr{B}(\mathscr{H})$$

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Example: Multiplication Representation

If (X, Ω, μ) is a σ-finite measure space, $L^\infty(X, \Omega, \mu)$ with complex conjugation as the *-operation is a C^*-algebra. The map

$$M : L^\infty(X, \Omega, \mu) \to \mathscr{B}(L^2(X, \Omega, \mu)),\ M_g(f) = gf, \forall g \in L^\infty(X, \Omega, \mu)$$

is a $*$-representation of $L^\infty(X, \Omega, \mu)$

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States of a C^*-algebra

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Def^n: State

A **state** $\varphi \in \mathscr{B}(\mathfrak{A}, \mathbb{C})$ is a bounded linear functional such that

$$||\varphi|| = 1, \quad \text{and} \quad \forall A \in \mathfrak{A}, \varphi(A^*A) \geq 0$$

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Notation: We write $\mathscr{S}(\mathfrak{A})$ for the **state space** of $\mathfrak{A}$

Concrete Model for a State: Construction

Thm: Representability of States

If $\rho \in \mathscr{S}(\mathfrak{A})$ there exists a representation $\varphi_\rho : \mathfrak{A} \to \mathscr{H}_\rho$ and $\xi_\rho \in \mathscr{H}_\rho$ such that

$$\mathrm{cl}(\varphi_\rho(\mathfrak{A})\xi_\rho) = \mathscr{H}_\rho \quad \text{and} \quad \rho(A) = \langle \varphi_\rho(A)\xi_\rho, \xi_\rho \rangle_{\mathscr{H}_\rho}, \ \forall A \in \mathfrak{A}$$

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Construction Sketch: ρ induces a semi-inner product u_ρ on $\mathfrak{A}$:

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Construction: State Space

To obtain an inner product let $\mathfrak{K} := \{A \in \mathfrak{A} : u_\rho(A, A) = 0\}$, and set $\mathfrak{B} = \mathfrak{A}/\mathfrak{K}$:

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- If $A \in \mathfrak{K}, C \in \mathfrak{A}$,

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Set $\mathscr{H}_\rho := \widehat{\mathfrak{B}}$, the Hilbert space completion of $\mathfrak{B}$.

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Construction: Action

Let $I \in \mathfrak{A}$ be the unit, and set $\xi_\rho := (I + \mathfrak{K})_{n\in\mathbb{N}}$.

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- $\varphi_\rho(\mathfrak{A})\xi_\rho = \mathfrak{A}/\mathfrak{K}$ embedded into $\widehat{\mathfrak{B}}$, which is dense.

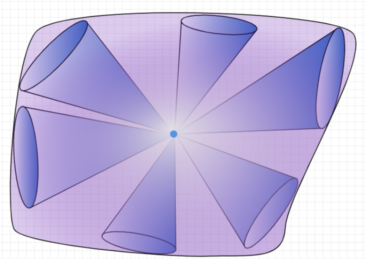
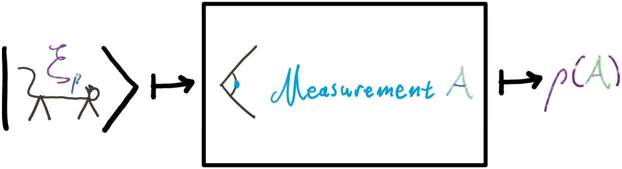


Figure: Visualization of $\varphi_\rho(\mathfrak{A})\xi_\rho$ in $\widehat{\mathfrak{B}} = \mathscr{H}_\rho$.

Construction: Measurement

A measurement in $\mathscr{H}_\rho$ is an evaluation of ρ:

$$\begin{aligned}\langle \varphi_\rho(A)\xi_\rho, \xi_\rho\rangle_{\mathscr{H}_\rho} &= \langle (A+\mathfrak{K})_{n\in\mathbb{N}}, (I+\mathfrak{K})_{n\in\mathbb{N}}\rangle_{\mathscr{H}_\rho} \\ &:= \lim_{n\to\infty} \langle A+\mathfrak{K}, I+\mathfrak{K}\rangle_{\mathfrak{B}} \\ &= \lim_{n\to\infty} \rho(A) = \rho(A)\end{aligned}$$



Conclusions

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- Quantum mechanics can be modeled by algebras of operators
- Operator algebras separate general observables and particular physical systems
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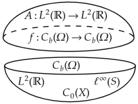

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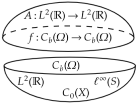

$$\mathfrak{A} \curvearrowright \mathscr{H}$$

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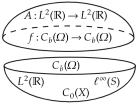

$\mathfrak{A} \curvearrowright \mathscr{H}$

Concluding Question: Can we model an algebra of operators faithfully, and if so how would we go about it?

Questions?

Thank you for your time!

Any questions?

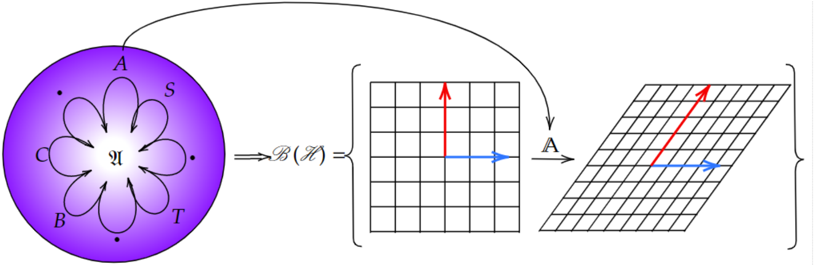

Figure: Diagram of C^*-algebra representation

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