A Glimpse into Categorical Logic

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Sum-C Presentation Seminar

Roadmap

$A \to B$
$B \leftarrow C$
$C \to A$

The What and the Why - 5 Slides

- What does it mean to categorify formal systems?
- Why do we want to do it?

The Basic Construction - 4 Slides

$$\mathbf{C}(A \wedge B, C) \cong \mathbf{C}(A, B \to C)$$

- What is a CCC?
- Adjunctions
- Inference Rules

The General Study and LL - 1 Slides

$$\frac{A \otimes B \vdash C}{A \vdash B \multimap C}$$

- Extensions to other systems

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What is a Formal System?

Def^n

A formal system[1] Γ consists of the following data [5]:

1. A collection of distinct symbols (the alphabet): e.g.

$$(,), \to, \wedge, \vee, A, B, C, ..., A_1, B_1, ...$$

2. A "grammar" for constructing well-formed formulas (wffs) of the language
3. A subcollection Λ of wffs called axioms: e.g.

$$\vdash A \to A, \vdash \top, \vdash A \to \top \vdash A \wedge B \to A, ...$$

4. Rules of inference on wffs: e.g. Modus Ponens, Modus Tollens, Disjunctive syllogism, etc.

[1] For alternative definitions see [1][11]

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What is a Category?

Def^n

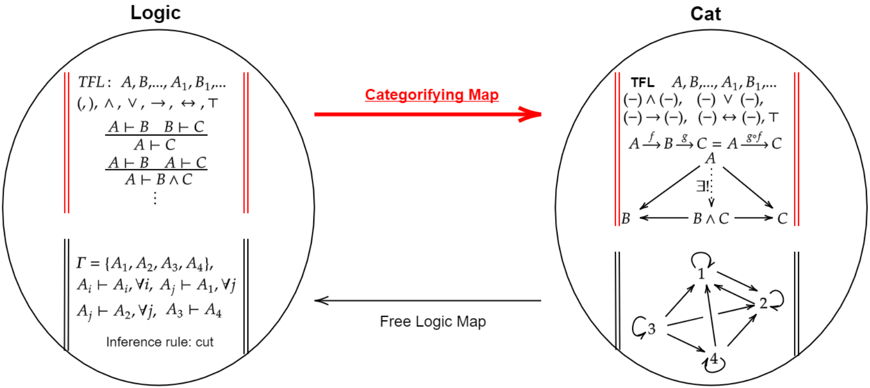
A Category $\mathbf{C}$ consists of the following data:

1. A class of objects $\mathbf{Ob}(\mathbf{C})$
2. $\forall A, B \in \mathbf{Ob}(\mathbf{C})$, a class $\mathbf{C}(A, B)$ of arrows
3. $\forall A \in \mathbf{Ob}(\mathbf{C})$, $\exists$ a distinguished arrow $\mathbf{id}_A \in \mathbf{C}(A, A)$

$$\begin{tikzcd}
A \arrow[loop left, "\mathbf{id}_A"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[loop right, "\mathbf{id}_B"] \arrow[d, "g"] \\
 & C \arrow[loop right, "\mathbf{id}_C"]
\end{tikzcd}$$

Category Theory resources [20][12][21]

The Categorification



Why Categorify?

Logical Notions

- Propositions

$\Longleftrightarrow$

Categorical Notions

- Objects

[2] For a formal definition see [18]
[3] For a formal definition see [17]

Why Categorify?

Logical Notions

- Propositions
- Proofs

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Categorical Notions

- Objects
- Arrows

[2] For a formal definition see [18]

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Why Categorify?

Logical Notions

- Propositions
- Proofs
- Inference Rules (e.g. Cut, Currying, Pairing)

$\Longleftrightarrow$

Categorical Notions

- Objects
- Arrows
- Methods of combining arrows (e.g. Natural Transformations[2])

[2] For a formal definition see [18]

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Why Categorify?

Logical Notions

- Propositions
- Proofs
- Inference Rules (e.g. Cut, Currying, Pairing)
- Models

$\Longleftrightarrow$

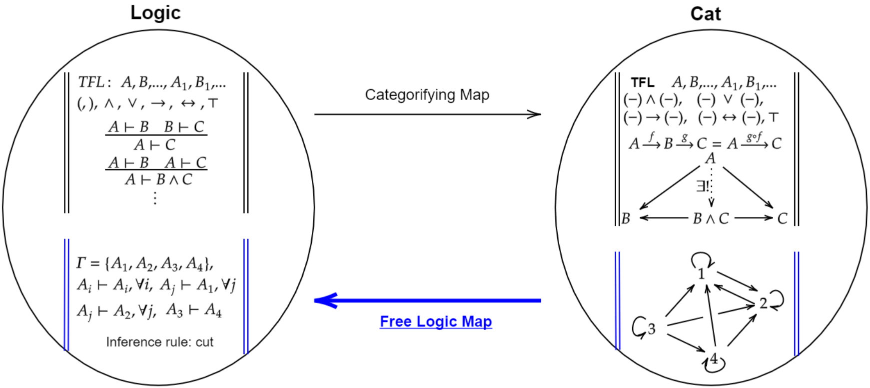
Categorical Notions

- Objects
- Arrows
- Methods of combining arrows (e.g. Natural Transformations[2])
- Certain structure preserving Functors[3]

[2] For a formal definition see [18]

[3] For a formal definition see [17]

The Benefits



What is a CCC?

Def^n

A cartesian closed category (ccc) **C** is a category with finite products $A \times_{\mathbf{C}} B$ and internal homs $[A, B]$, $\forall A, B \in \mathbf{ob}(\mathbf{C})$, and for any $B \in \mathbf{Ob}(\mathbf{C})$, a pair of functors

$$(-) \times_{\mathbf{C}} B : \mathbf{C} \to \mathbf{C}$$

and

$$[B, -] : \mathbf{C} \to \mathbf{C}$$

which sends an object A to its product with B and its internal homset out of B, respectively.[4]

[4] For a more complete definition see [14]

Adjoint Functors

Def^n

We say $\mathscr{A} \underset{\mathscr{G}}{\overset{\mathscr{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{B}$ are adjoint functors if for any $A \in \mathbf{Ob}(\mathscr{A}), B \in \mathbf{Ob}(\mathscr{B})$, they induce an isomorphism

$$\mathscr{B}(\mathscr{F}(A), B) \cong \mathscr{A}(A, \mathscr{G}(B))$$

which is natural[5] in A and B.

- In our ccc $\mathbf{C}$, we have

$$\mathbf{C}(A \times_{\mathbf{C}} B, C) \cong \mathbf{C}(A, [B, C])$$

for all $A, B, C \in \mathbf{Ob}(\mathbf{C})$, and write

$$\frac{A \times_{\mathbf{C}} B \xrightarrow{f} C}{A \xrightarrow{\overline{f}} [B, C]}$$

[5] For a more complete definition see [13]

Inference Rules

- $\mathbf{C}(A \wedge B, C) \cong \mathbf{C}(A, B \to C)$:

Currying Rule

$$\frac{A \xrightarrow{f} (B \to C)}{(A \wedge B) \xrightarrow{\overline{f}} C}$$

$$\Updownarrow$$

$$\overline{A \xrightarrow{f} (B \to C)} = (A \wedge B) \xrightarrow{\overline{f}} C$$

- Modus Ponens:

Counit[6]

$$B, B \to A \vdash A$$

$$\Updownarrow$$

$$[(B \to A) \wedge B] \xrightarrow{\varepsilon_A} A = \overline{(B \to A) \xrightarrow{\mathrm{id}_{B \to A}} (B \to A)}$$

[6] For a definition see [19]

Inference Rules (cont.)

Coproduct[7]

$$
\begin{tikzcd}
 & \mathscr{C} & \\
\mathscr{A} \arrow[ur, "p_1"] \arrow[r, "\iota_{\mathscr{A}}"'] & \mathscr{A} \vee \mathscr{B} \arrow[u, dashed, "\exists! p"'] & \mathscr{B} \arrow[ul, "p_2"'] \arrow[l, "\iota_{\mathscr{B}}"]
\end{tikzcd}
$$

$\iff$

Disjunctive Elimination

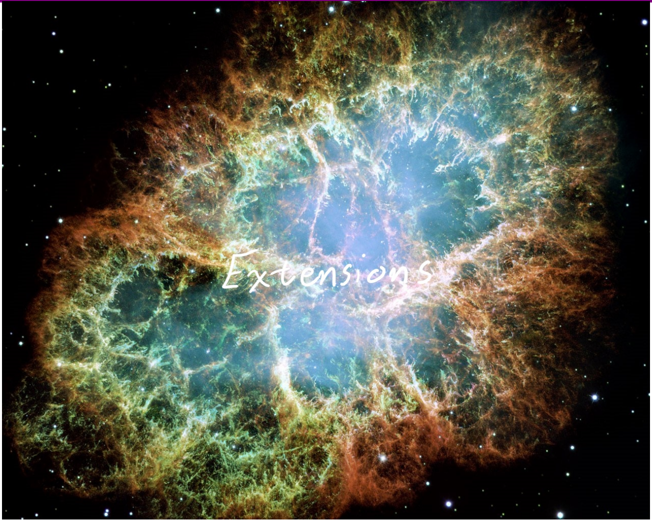
Axioms:

$$\mathscr{A} \vdash_{\iota_{\mathscr{A}}} \mathscr{A} \vee \mathscr{B} \quad \text{and} \quad \mathscr{B} \vdash_{\iota_{\mathscr{A}}} \mathscr{A} \vee \mathscr{B}$$

Inference Rule:

$$\frac{\mathscr{A} \vdash_{p_1} \mathscr{C} \qquad \mathscr{B} \vdash_{p_2} \mathscr{C}}{\mathscr{A} \vee \mathscr{B} \vdash_{p} \mathscr{C}}$$

[7] For a full definition see [16]



Linear Logic resources [4][3][9][10]
Image [6]

Summary

1. **Categorify**:
$$\begin{aligned} Propositions &\iff Objects \\ Proofs &\iff Arrows \\ Inference\ Rules &\iff Arrow\ Transformations \end{aligned}$$

[8] See [4]
[9] For a full definition see [15]

Summary

1. **Categorify**:

$$\begin{array}{rcl} Propositions & \iff & Objects \\ Proofs & \iff & Arrows \\ Inference\ Rules & \iff & Arrow\ Transformations \end{array}$$

2. **CCC and Adjunctions**:

$$\begin{array}{c} (-) \times_{\mathbf{C}} B : \mathbf{C} \to \mathbf{C} \\ [B, -] : \mathbf{C} \to \mathbf{C} \end{array} \implies \begin{array}{c} \mathbf{C}(A \times_{\mathbf{C}} B, C) \cong \\ \mathbf{C}(A, [B, C]) \end{array}$$

[8] See [4]

[9] For a full definition see [15]

Summary

1. **Categorify**:
$$\begin{aligned} \mathit{Propositions} &\iff \mathit{Objects} \\ \mathit{Proofs} &\iff \mathit{Arrows} \\ \mathit{Inference\ Rules} &\iff \mathit{Arrow\ Transformations} \end{aligned}$$

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$$\begin{matrix} (-) \times_{\mathbf{C}} B : \mathbf{C} \to \mathbf{C} \\ [B, -] : \mathbf{C} \to \mathbf{C} \end{matrix} \implies \begin{matrix} \mathbf{C}(A \times_{\mathbf{C}} B, C) \cong \\ \mathbf{C}(A, [B, C]) \end{matrix}$$

3. **Inference Rules**:

Pairing[8]

$$\frac{\mathscr{C} \vdash_{f_1} \mathscr{A} \qquad \mathscr{C} \vdash_{f_2} \mathscr{B}}{\mathscr{C} \vdash_{f} \mathscr{A} \wedge \mathscr{B}}$$

$\iff$

Products[9]

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists! f"'] \arrow[dr, "f_2"] & \\ \mathscr{A} & \mathscr{A} \wedge \mathscr{B} \arrow[l, "\pi_{\mathscr{A}}"] \arrow[r, "\pi_{\mathscr{B}}"'] & \mathscr{B} \end{tikzcd}$$

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4. **Extensions**: Linear Logic

[8] See [4]
[9] For a full definition see [15]

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