Let $\mathcal{B}$ be an ∞-topos.

Definition ($\mathcal{B}$-Category and $\mathcal{B}$-functors)

A $\mathcal{B}$-category $\mathcal{C}$ is a limit preserving functor $\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Cat}_\infty$. Write $\mathrm{Cat}(\mathcal{B}) = \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ for the (large) category of $\mathcal{B}$-categories. A $\mathcal{B}$-functor is a natural transformation of $\mathcal{B}$-categories, and we will write $\mathrm{Fun}_{\mathcal{B}}(\mathcal{C}, \mathcal{D}) := \mathrm{Nat}(\mathcal{C}, \mathcal{D})$ for the category of $\mathcal{B}$-functors.

Important

Regular (higher) category theory has $\mathcal{B} = \mathrm{Spc} = \mathrm{PShv}(*)$ since $\mathrm{Fun}^{\mathrm{R}}(\mathrm{PShv}(*)^{\mathrm{op}}, \mathrm{Cat}_\infty) \simeq \mathrm{Fun}(*^{\mathrm{op}}, \mathrm{Cat}_\infty) = \mathrm{Cat}_\infty$. More generally, $\mathrm{PShv}(\mathrm{T})$-categories are just Cat_∞-valued presheaves on T.

Definition (Underlying Category)

Given a $\mathcal{B}$-category $\mathcal{C}$, we will call the category, $\Gamma\mathcal{C} := \mathcal{C}(1)$, given by evaluating on the terminal object $1 \in \mathcal{B}$ the underlying category of $\mathcal{C}$.

Example ($\mathcal{B}$-groupoid)

We have the Yoneda embedding $h_{(-)} : \mathcal{B} \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Spc}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ giving for each $B \in \mathcal{B}$ a $\mathcal{B}$-category $\underline{B} = \mathrm{hom}_{\mathcal{B}}(-, B)$. Such $\mathcal{B}$-categories are called $\mathcal{B}$-groupoids.

Example ($\mathcal{B}$-groupoids)

The functor $\mathrm{ev}_0 : \mathcal{B}^{\Delta^1} \to \mathcal{B}$ is a cartesian fibration and hence classifies a limit-preserving functor

$$\Omega_{\mathcal{B}} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad B \mapsto \mathcal{B}_{/B}, \quad f \mapsto f^*$$

which is a $\mathcal{B}$-category called the $\mathcal{B}$-category of $\mathcal{B}$-groupoids.

Example (Functor $\mathcal{B}$-category)

The ∞-category $\mathrm{Cat}(\mathcal{B})$ is cartesian closed and thus has an internal hom $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ (we will supress the $\mathcal{B}$ in the subscript when $\mathcal{B}$ is clear from context) which satisfies $\Gamma\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$.

Analogies with Normal Category Theory

(Paremetrized Yoneda Lemma)

For each $B \in \mathcal{B}$, evaluation at $\mathrm{id}_B \in \underline{B}(B)$ defines a natural equivalence

$$\mathrm{Fun}_{\mathcal{B}}(\underline{B}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}(B).$$

As a consequence there are natural equivalences

$$\underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{B}, \mathcal{C}) \simeq \mathcal{C}(B \times (-))$$

and

$$\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{C} \times \underline{B}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{C}, \underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{B}, \mathcal{D})).$$

Definition ($\mathcal{Q}$-cocomplete)

Let $\mathcal{Q}$ be a class of morphisms in $\mathcal{B}$ closed under base change. A $\mathcal{B}$-category $\mathcal{C}$ is $\mathcal{Q}$-complete if it satisfies the following conditions:

- for every $f : A \to B$ in $\mathcal{Q}$, the functor $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $f_! : \mathcal{C}(A) \to \mathcal{C}(B)$
- For every pullback square,

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{\beta} & B \end{array}$$

  in $\mathcal{B}$ with f in $\mathcal{Q}$, the Beck-Chevalley transofmration $f'_! \alpha^* \Rightarrow \beta^* f_!$ is an equivalence.

We will typically assume that $\mathcal{Q}$ is *local* meaning that a morphism $f : A \to B$ is in $\mathcal{Q}$ whenever there exists an effective epimorphism $\bigsqcup_{i \in I} B_i \twoheadrightarrow B$ in $\mathcal{B}$ such that each of the base change maps $A \times_{\mathcal{B}} B_i \to B_i$ is in $\mathcal{Q}$.

Definition (Presentable $\mathcal{B}$-category)

A $\mathcal{B}$-category $\mathcal{C}$ is presentable if it is fiberwise presentable, that is it factors through Pr^{L}, and it is $\mathcal{B}$-cocomplete.

Example

Since $\mathcal{B}$ is an ∞-topos, $\Omega_{\mathcal{B}}$ is fibre-wise presentable and since $\mathrm{ev}_0 : \mathcal{B}^{\Delta^1} \to \mathcal{B}$ is a *Beck-Chevalley fibration*, as in Hopkins-Lurie, then $\Omega_{\mathcal{B}}$ is also presentable.

Definition

A $\mathcal{B}$-functor $F : \mathcal{C} \to \mathcal{D}$ between presentable $\mathcal{B}$-categories preserves (parametrized) colimits if

- For each $B \in \mathcal{B}$, the functor $F(B) : \mathcal{C}(B) \to \mathcal{D}(B)$ preserves colimits,
- For every morphism $f : A \to B \in \mathcal{B}$, the Beck-Chevalley tranformation $f_! \circ F(A) \Rightarrow F(B) \circ f_!$ is an equivalence.
  We will write $\mathrm{Fun}^{\mathrm{L}}_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ for the category of colimit-preserving $\mathcal{B}$-functors.

Definition

Let $\mathrm{Pr}^{\mathrm{L}}(B)$ denote the category of presentable $\mathcal{B}$-categories with colimit-preserving $\mathcal{B}$-functors between them.

Ambidexterity

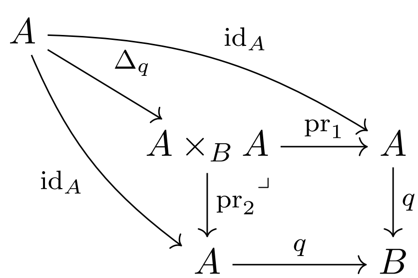
Definition ((Locally) Inductible)

A wide local subcategory $\mathcal{Q} \subseteq \mathcal{B}$ closed under base change is locally inductible if

- every morphism $q\colon A \to B$ in $\mathcal{Q}$ locally truncated: there exists a covering $(B_i \to B)_{i \in I}$ (i.e. the induced map $\bigsqcup_{i \in I} B_i \to B$ is an effective epimorphism) such that each base change $q_i\colon B_i \times_B A \to B_i$ is truncated, and
- $\mathcal{Q}$ is closed under diagonals: every morphism $q : A \to B$ in $\mathcal{Q}$, the diagonal map $\Delta_q : A \to A \times_B A$ is again in $\mathcal{Q}$.
  Furthermore, if every morphism in $\mathcal{Q}$ is truncated, we'll say $\mathcal{Q}$ is inductible.

Let Q be a locally inductible subcategory of $\mathcal{B}$ and $\mathcal{C}$ be a $\mathcal{Q}$-cocomplete $\mathcal{B}$-category, then the restriction of $\mathcal{C}$ to $\mathcal{Q}^{\mathrm{op}}$ can be unstraightened to a Beck-Chevalley fibration, $\int(\mathcal{C}|_{\mathcal{Q}^{\mathrm{op}}}) \to \mathcal{Q}$. In this setup, we will define what it means for an n-truncated morphism $q \in Q$ to be $\mathcal{C}$-ambidextrous, which will allow us to construct a map $\mu_q^{(n)} : \mathrm{id}_{\mathcal{C}(B)} \to q_!q^*$ exhibiting $q_!$ as right adjoint to q^*.

The induction starts at $n = -2$, where q is an equivalence and is declared to be $\mathcal{C}$-ambidextrous. Since q is an equivalence, the counit map $q_!q^* \to \mathrm{id}_{\mathcal{C}(B)}$ is an equivalence and $\mu_q^{(-2)} : \mathrm{id}_{\mathcal{C}(B)} \to q_!q^*$ is defined as the inverse.
Assume now that we have defined what it means for an n-truncated morphism to be $\mathcal{C}$-ambidextrous for some $n \geq -2$ with the required transformations $\mu_{(-)}^{(n)}$. In this case, we say that an $(n+1)$-truncated morphism $q : A \to B$ is *weakly $\mathcal{C}$-ambidextrous* if its diagonal $\Delta_q : A \to A \times_B A$ is $\mathcal{C}$-ambidextrous (which is well-defined since Δ_q is n-truncated). Then from the following commutative diagram



Definition (Norm map)

If $q : A \to B$ in $\mathcal{Q}$ is n-truncated and $\mathcal{C}$-ambidextrous. Then the norm map $\mathrm{Nm}_q : q_! \xrightarrow{\sim} q_*$ is defined to be the composition

$$q_! \xrightarrow{c_q^* \circ q_!} q_*q^*q_! \xrightarrow{q_* \circ \widetilde{\mathrm{Nm}}_q} q_*.$$

Definition ($\mathcal{Q}$-semiadditive)

For $\mathcal{Q}$ a locally inductible category, a $\mathcal{Q}$-cocomplete $\mathcal{B}$-category $\mathcal{C}$ is called $\mathcal{Q}$-semiadditive every truncated morphism $q : A \to B$ in $\mathcal{Q}$ is $\mathcal{C}$-ambidextrous.

Remark

Note that even if we cannot talk about an non-truncated $q : A \to B$ in $\mathcal{Q}$ being $\mathcal{C}$-ambidextrous, the local inductibility condition on $\mathcal{Q}$ allows us to construct a norm map $\mathrm{Nm}_q : q_! \to q_*$ anyways.

Examples

To talk about examples that we know, it'll be convenient to introduce a new notion of inductible subcategory in the case of $\mathcal{B} = \mathrm{PShv}(T)$ being a presheaf topos.

Definition (Pre-inductible subcategory)

Given a replete subcategory $\mathcal{Q} \subseteq \mathrm{PShv}(T)$ containing all representables, we say it is pre-inductible if the following holds:

- given a morphism $q : A \to B$ in $\mathrm{PShv}(T)$ with $B \in \mathcal{Q}$. Then q lies in $\mathcal{Q}$ if and only if for each pullback square in $\mathrm{PShv}(T)$ with $t \in T$, the base change q' also lies in $\mathcal{Q}$.
- $\mathcal{Q}$ is closed under diagonals.
- Every morphism in $\mathcal{Q}$ with target in T is truncated.

Definition (Locally Inductible Subcategory Generated by $\mathcal{Q}$)

Let $\mathcal{Q} \subseteq \mathrm{PShv}(T)$ be a pre-inductible subcategory. We say $q : A \to B$ is *locally* in $\mathcal{Q}$ if for every morphism $B' \to B$ in $\mathrm{PShv}(T)$ with $B' \in \mathcal{Q}$, the base change $A \times_B B' \to B'$ lies in $\mathcal{Q}$. Since these morphisms are closed under composition and contain all equivalences, they determine a wide subcategory, denoted $\mathcal{Q}_{\mathrm{loc}}$, called the locally inductible subcategory generated by $\mathcal{Q}$.

Lemma

As a subcategory of $\mathrm{PShv}(T)$, $\mathcal{Q}_{\mathrm{loc}}$ is locally inductible.

Definition ($\mathcal{Q}$-semiadditivity again)

For a pre-inductible subcategory $\mathcal{Q} \subseteq \mathrm{PShv}(T)$, we say a $\mathrm{PShv}(T)$-category $\mathcal{C}$ is $\mathcal{Q}$-semiadditive if it is $\mathcal{Q}_{\mathrm{loc}}$-semiadditive.

Examples

T	$\mathcal{Q}$	$\mathcal{Q}$-semadditivity
$*$	Spc_m $(-2 \leq m \leq \infty)$	m-semiadditivity
$*$	$\mathrm{Spc}_m^{(p)}$ $(-2 \leq m \leq \infty)$	p-typical m-semiadditivity
Orb_G	Fin_G	G-semiadditivity
Glo	$\mathrm{FinGrpd}_{\mathrm{faithful}}$	Equivariant semiadditivity
Glo	$\mathrm{FinGrpd}$	Global semiadditivity

Here, $\mathrm{Glo} \subset \mathrm{FinGrpd}$ is the subcategory of connected finite groupoids.

Supposedly an Example

Given a six functor formalism $\mathcal{D}$, if we only remember the contravariant functorality, then we have functor $T^{\mathrm{op}} \to \mathrm{Cat}_\infty$, whose limit preserving extension is a $\mathrm{PShv}(T)$-category that is $\mathcal{Q}$-semiadditive with respect to the pre-inductible subcategory $\mathcal{Q}$ whose morphisms consist of cohomologically proper and étale morphisms in T.

Another Fun Example

If $\mathcal{T} \subset \mathbf{Glo}$ is the full subcategory spanned by groupoids with abelian isotropy, then the $\mathbf{PShv}(\mathcal{T})$-category of tempered local systems associated to an oriented $\mathbf{P}$-divisible group is $R(\mathrm{Spc}_\pi)$-semiadditive, where here $R : \mathrm{Spc} \to \mathrm{PShv}(T)$ is the fully faithful right adjoint of $\mathbf{ev}_* : \mathrm{PShv}(T) \to \mathrm{Spc}$.

Twisted Ambidexterity

(WIP)

(Classificiation of $\mathcal{C}$-linear Functors)

Given a C