Resource Theories for Random Discrete Dynamical Systems

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Abstract

Random discrete dynamical systems have a wide variety of applications, including complex systems and genetic regulation. However, many of these dynamical systems are non-isolated, and current approaches to external influences either use randomness to describe the presence of an external influence or only consider small external influences. In [1], Scandolo et al. introduced a theory for working with dynamical systems subject to a special class of external influences, called covariant influences, and fully characterized transitions in the deterministic case. In this thesis we provide a number of powerful results extending this work to the stochastic setting (i.e. for probabilistic transitions): we provide a complete description of stochastic transitions between systems that only have dynamical cycles, analyze constraints on transition probabilities, and provide necessary and sufficient conditions for the existence of influences which induce stochastic transitions between arbitrary states. The methodology for this work combines the abstract theory of resources with methods from linear algebra and graph theory to provide a general framework which applies to dynamical systems independently of their unperturbed evolution.

Keywords: dynamical systems, resource theories, matrix theory

1. Introduction

Dynamical systems are ubiquitous throughout the physical sciences, appearing in fields such as biological modelling [2], complex systems theory in statistical mechanics [3], physics and chaos theory [4], up to neuroscience [5]. Classically, two central varieties of dynamical systems have been studied: dynamical systems corresponding to continuous time evolutions, and dynamical systems corresponding to discrete time evolutions. In this thesis we restrict to the case of discrete dynamical systems. In either setting, the study of dynamical systems often uses perturbative methods which allow the systems to be studied while subjected to sufficiently small external influences. Two major gaps in the current theory are in the extension from small external influences to all external influences and frequent absence of frameworks that can answer questions about a large class of systems without solving their dynamics.

To begin filling this gap, in [1] Scandolo, Gour, and Sanders introduced for the first time the notion of covariant influences on discrete dynamical systems. Covariant influences model external effects which act on sufficiently long time scales so that the evolution of the system is not significantly disturbed. Due to this added structure, we are then able to consider covariant influences which are beyond the small perturbation regime.

The construction in [1] approaches such long-timescale influences using the powerful framework of resource theories, originally developed in quantum information theory [6]. Resource theories were originally investigated to understand how objects in quantum information theory [6], such as quantum entanglement [7], could be understood as a resource for communication or other quantum protocols. In 2016 Coecke et al. [8] formalized resource theories, providing a rigorous foundation for studying issues of conversion between resources using the mathematical field of category theory. Specifically, resource theories

focus on the study of operations between resources which can be performed at no cost. Due to this, a primary goal when initially studying any new resource theory is a solution to the conversion problem, which asks if, given two resources, one can be converted into the other at no cost. A simple example of a resource theory is given by considering combinations of molecules as resources, with certain chemical reactions as the method of conversion [9].

In the case of the resource theory constructed in [1], the conversion problem equates to asking when a fixed state in a dynamical system can be converted into another fixed state of a possibly different dynamical system via a covariant influence. This problem comes in two flavours, one where the covariant influence is deterministic and one where the covariant influence is stochastic. In [1], Scandolo et al. determined a finite set of constraints which provide necessary and sufficient conditions for the conversion problem in the deterministic setting. However, Scandolo et al. also showed in [1] that some of the constraints break down when randomness is introduced into the system in the form of stochasticity of the covariant influences.

In this thesis we expand on the results in [1] by determining constraints on stochastic covariant influences that appear in the case of random discrete dynamical systems (RDDS) with a finite number of states. This analysis is performed using a variety of tools from the theory of resources, stochastic matrix theory, and graph theory. Resource theories are used for providing a foundational framework for the work, as well as to produce high-level results on the interaction between covariant influences and disjoint components of a RDDS. On the other hand, the main computational work is done using stochastic matrix theory. This process involves the consideration of a special class of matrices which allow for a clean description of the interaction between covariant influences and the cycles that appear in finite dynamical systems. Finally, all results are interpreted using graphical aspects of the

dynamical systems in order to develop an efficient graphical calculus for practical use.

In Sec. 2 we begin by introducing the appropriate background on finite dynamical systems which is necessary to understand the work. We also summarize the results of [1] in the context of the current work, emphasizing graphical intuition. Following this overview, in Sec. 3 we introduce the details of the primary methods used in this thesis. First, we describe resource theories, with an emphasis on intuition and analogies to account for the complexity and abstraction involved in their original definition in [8]. Secondly, we introduce the theory of circulant matrices and how they naturally appear in the study of covariant influences on finite dynamical systems. Following these foundations, in Sec. 4 we describe the main results obtained during the thesis project including a complete description of covariant influences in cycles, upper bounds on transition probabilities, and the wealth of constraints that appear when cycle-only dynamics are no longer assumed. In Sec. 5 we conclude with a summary of the primary results obtained and possible future directions for the work performed in the thesis.

2. Background

Mathematically, discrete dynamical systems have a simple and elegant formulation: A discrete dynamical system on a set of states S is a function $\phi : S \to S$ assigning an output state to every input state [10]. The action of ϕ on states represents an evolution of the states over a single time step. To obtain the evolution of the system after n times steps, for $n \in \mathbb{N} = \{0, 1, 2, ...\}$, we simply apply ϕ n times. This setup can also be visualized using a directed graph, called the *dynamical graph* of the system, as in Fig. 1, which represents the system's state space. A directed graph consists of a pair (V, E) where V is the set of vertices or nodes and E is the set of edges between vertices with an indicated direction. For a dynamical graph, nodes represent states in the dynamical system, while directed

arrows indicate the evolution of the system in a single time step.

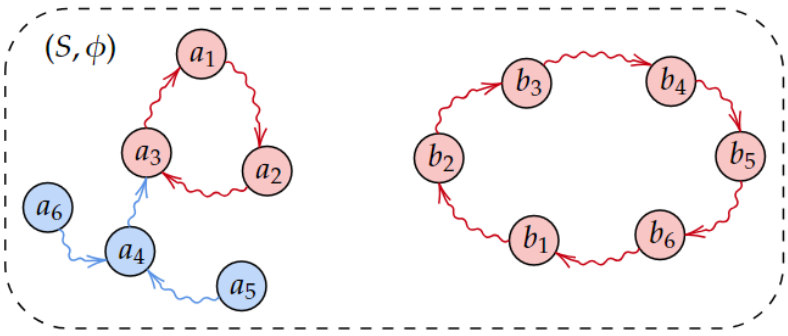

Figure 1. Dynamical system (S, ϕ) with two basins, where transient states are drawn blue and attractor states are drawn red. The dynamics for the "a" basin branches into two directions at the state a_4.

There are a number of important features of dynamical systems which are quickly illuminated by their dynamical graphs [1]. First, discrete dynamical systems can be partitioned into basins of attraction which correspond to distinct connected subgraphs of the system's dynamical graph. Two such basins are seen for the dynamical graph in Fig. 1. For finite dynamical systems, each basin contains a cycle called the *attractor* for the basin, and within which the dynamics evolve cyclically (e.g. red states in Fig. 1). The number of states in such a cycle is called the *length* of the basin it is contained within [1]. We say that states which are not part of a cycle are *transient* (e.g. blue states in Fig. 1), noting that all transient states eventually transition into a cycle given a sufficient number of time steps.

As in [1], the degree to which a state s is transient is measured by two properties, its *transient progeny*, $d(s)$, and its *ancestry*, $a_0(s)$. The transient progeny of s gives the number of time steps required for it to arrive in a cycle, while the ancestry gives the largest integer k for which an ancestor s' of s exists such that $\phi^k(s') = s$. By convention we say that $a_0(s) = +\infty$ if s is a state in a cycle. To describe ancestry graphically we introduce the notion of a *parent* of a state. The parent of a state s is a state s' such that $\phi(s') = s$. We

say that a state is *parentless* if it has no parents (e.g. states a_5 and a_6 in Fig. 1).

In the theory of cellular regulation, finite dynamical systems have been used to model gene activation and regulation in cells [11]. Explicitly, the states of the dynamical system in this case are strings of 0s and 1s indicating which genes in a given cell are expressed or suppressed. In this setting, one of the theoretical predictions is that distinct cycles represent distinct cell types [11].

Applications of dynamical systems, such as in genetic regulation [11], often need to consider interactions with the system's external environment which is not accounted for in the dynamical systems' definition. This problem was solved in genetic regulation theory by introducing randomness into the model through the replacement of states by probability vectors and the replacement of the deterministic dynamics by stochastic dynamics [12]. This approach has lead to many important results in gene theory, including the "cell types as cycles" paradigm as well as the fact that the number of cell types follow a power law with respect to the number of genes [11].

For other non-isolated systems, external effects are considered as small perturbations to the standard dynamics of the system [13]. In [1], Scandolo et al. introduced the notion of covariant influences on discrete dynamical influences in order to rigorously study external effects beyond the small perturbation regime. In the deterministic setting, a covariant influence on a dynamical system (S, ϕ) is a function $F : S \to S$ on the state space which commutes with the dynamics, ϕ, in the sense that $F(\phi(s)) = \phi(F(s))$ for any state $s \in S$. Physically, a covariant influence represents an external influence which acts over a sufficiently long time scale so as not to drastically disturb the system dynamics.

Scandolo et al. formulated this new model for external influences on dynamical systems in

both deterministic and stochastic settings in [1]. In the case of deterministic influences, they fully characterized possible state transitions in terms of features of the system's dynamical graph. Explicitly, they proved that for states s and s' of S, where s is in a basin of length ℓ and s' is in a basin of length ℓ', a covariant influence on F which sends s to s' exists if and only if the following conditions hold:

$$a_0(\phi^k(s)) \leq a_0(\phi^k(s')) \quad \text{for} \quad k = 0, 1, ..., d(s') - 1, \;\; d(s) \geq d(s'), \quad \text{and} \quad \ell' \;\; \text{divides} \;\; \ell \quad (1)$$

The first two conditions say that s' and its descendants must be closer to the attractor of its basin than s, while the last condition states that in the time the input cycle is traversed once, the output cycle must be traversed a whole number of times.

In the stochastic setting, functions describing covariant influences, $F : S \to S$, become stochastic matrices acting on probability vectors over the state space S. Here a stochastic matrix is a matrix with entries between 0 and 1 representing transition probabilities, where the entries in a given column sum to 1. Physically this says that a deterministic state is sent to a probability vector of states. Moving forward we write the set of probability vectors on the state space S using caligraphic notation as $\mathcal{S}$, and we also write the linear map on probability vectors induced by ϕ by capitalizing $\Phi : \mathcal{S} \to \mathcal{S}$. In this context a stochastic covariant influence is exactly a stochastic matrix $F : \mathcal{S} \to \mathcal{S}$ such that $\Phi F = F\Phi$. We denote the entries of F by $p_F(s'|s)$ for states s, s' in the underlying system, indicating that the entry gives the probability of transitioning from s to s' under the covariant influence F. Scandolo et al. showed that for state transitions with stochastic influences the divisibility condition in Eq. (1) breaks down [1]. However, a treatment that is able to elucidate the connection between different entries of the stochastic matrix as a result of the covariance condition was largely missing.

3. Methods

In order to construct a theory which is fully formal and can be applied to random dynamical systems in a fashion that is independent of the details of the system, Scandolo et al. exported the framework of resource theories beyond the domain of quantum information theory to model their covariant influences [1]. In this section we describe the details of resource theories in our current work at a high level in order to make the abstract concepts accessible to non-experts. We also present more concrete matrix algebra methods that are essential to understanding the stochastic covariant influences under consideration.

3.1. Resource Theories

Resource theories, which were first studied in quantum information theory [6], were expanded on by Coecke et al. [8] into a fully formal mathematical theory for studying the conversion between resources at no cost. A resource theory consists of two main pieces of data: an ambient process theory which models physical systems and physical processes that occur on them, and the specification of a collection of processes which occur at no cost, called free processes. In general processes between physical systems can occur sequentially or in parallel, and the same is true of processes which occur at no cost. In addition to physical systems and processes, we consider a more theoretical object which is the trivial system. The importance of the trivial system comes from the fact that processes from the trivial system into any other physical system correspond to preparing a state of the other system.

We can then consider a resource theory as a smaller process theory where all processes can be performed at no cost. The resources of a given theory are given by states of the physical systems being considered. The primary goal for any resource theory is then a solution to the conversion problem, which asks when given resources A and B whether a

free process from the system containing A to the system containing B exists which converts A into B. Intuitively, an answer to this question signifies whether A is a "more valuable" resource than B. A simple example of a resource theory is given by considering collections of molecules as physical systems with chemical reactions being the processes [9]. The free processes can then be those chemical reactions which occur at room temperature and standard pressure without the need for other catalysts.

In [1], Scandolo et al. defined a resource theory of random discrete dynamical systems and covariant influences. The ambient process theory consists of RDDSs with all possible stochastic influences between them. We write an influence between RDDSs (S, ϕ) and (T, ψ) by $F : \mathcal{S} \to \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are the stochastic counterparts of S and T, as in Sec. 2. The processes which occur at no cost for the theory are then those processes F which are also covariant influences, so $F\Phi = \Psi F$, for Φ and Ψ the matrices associated with ϕ and ψ.

3.2. Circulant Matrices

In this section we introduce a class of matrices which describe the transition blocks in a covariant influence between systems with no transient states. These are *circulant matrices*, which have a natural cyclic definition reflecting the cyclicity of attractors in finite dynamical systems.

An $m \times n$ circulant matrix A consists of a first row $(\alpha_1 \cdots \alpha_n)$, with each successive

row being a cyclic shift of the row above by one step to the left. Explicitly we can write

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_2 & \alpha_3 & \cdots & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_m & \alpha_{m+1} & \cdots & \alpha_{m+n-1} \end{pmatrix}, \tag{2}$$

where we compute the indices modulo n in the last row, and where we require that the first row is a shift to the left of the last row, so $(\alpha_{m+1} \cdots \alpha_{m+n}) = (\alpha_1 \cdots \alpha_n)$. Matrices describing permutations of states when $n = m$ are an important special case of these circulant matrices.

During the execution of the thesis it was determined that these circulant matrices form a subspace of $m \times n$ matrices of dimension given by the greatest common divisor of m and n, $\gcd(m, n)$. A basis of this subspace is given by $C^{(0)}_{m,n}, ..., C^{(\gcd(m,n)-1)}_{m,n}$, which are generalizations of permutation matrices. Explicitly, $C^{(i)}_{m,n}$ can be described as the circulant matrix with first row having 1s in the entries $i{+}1, i{+}\gcd(m, n){+}1, ..., i{+}(\ell{-}1)\gcd(m, n){+}1$, where ℓ is the integer for which $\ell \gcd(m, n) = n$. In the case of $m = 2$ and $n = 4$, such a basis is of the following form:

$$\left\{ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

This simple class of permutation-like matrices allow for a simple and elegant description of covariant influences on systems with no transient states.

4. Results and Discussion

The use of resource theories as an underlying framework allows us to develop a general theory of random discrete dynamical systems (RDDS) subject to long-timescale influences. We will begin by demonstrating how resource theories allow us to simplify the study of these influences by inductively constructing complicated systems with a large number of basins together from far simpler systems. This result will then allow us to develop a full characterization of covariant influences for cycle-only RDDSs before introducing transience.

4.1. Constructing Dynamical Systems

In practice RDDSs can consist of an extremely large number of basins of attraction, each with numerous states. However, as we show in this section, the covariance assumption guarantees that covariant influences are well-behaved with respect to basin-to-basin interactions. Explicitly, the transition probabilities for a covariant influence can be partitioned into blocks which represent transitions between two fixed basins at a time, greatly simplifying the problem of classifying these influences.

The primary fact that this result relies on is that a covariant influence out of a dynamical system is fully determined by specifying covariant influences out of each basin of attraction. Indeed, suppose we wanted to construct a stochastic covariant influence $F : (\mathcal{S}, \Phi) \to (\mathcal{T}, \Psi)$ between underlying systems (S, ϕ) and (T, ψ).

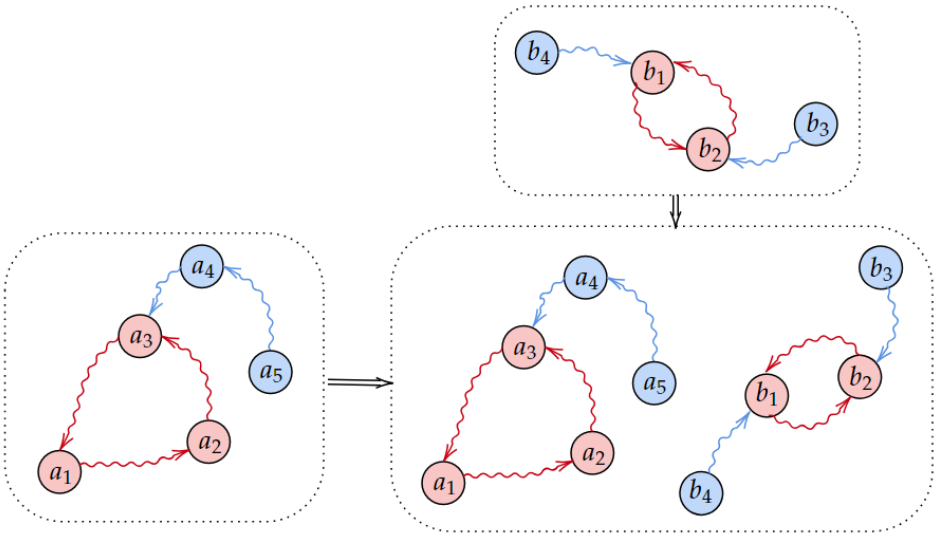

Figure 2. Diagram illustrating the construction a larger dynamical system from two disjoint dynamical systems.

Since (S, ϕ) is finite, it can be partitioned into a finite number of basins of attraction, $S_1, ..., S_n$, which are also the connected components for the dynamical graph of (S, ϕ). Note that we can restrict the dynamics to these subsystems to get dynamical systems $(S_1, \phi_1), ..., (S_n, \phi_n)$. We then have that (S, ϕ) is obtained by collecting these disjoint subsystems as in Fig. 2.

Then to specify F it is sufficient to give stochastic covariant influences $F_i : (\mathcal{S}_i, \Phi_i) \to (\mathcal{T}, \Psi)$ for each $i = 1, 2, ..., n$. Indeed, as a matrix F can be built out of the F_i by using our ordering on the basins of attraction in (S, ϕ), and writing F in block matrix form

$$F = \begin{pmatrix} F_1 & F_2 & \cdots & F_n \end{pmatrix}. \tag{3}$$

Since each F_i is stochastic so is F, as each column is a column of some F_i and hence a probability vector. On the other hand, the reason that F is covariant is due to the fact

that we can write Φ in block diagonal form in terms of the Φ_i as

$$\Phi = \begin{pmatrix} \Phi_1 & 0 & \cdots & 0 \\ 0 & \Phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_n \end{pmatrix}.$$

Using block multiplication of matrices and the covariance of the F_i this implies that

$$F\Phi = \begin{pmatrix} F_1\Phi_1 & \cdots & F_n\Phi_n \end{pmatrix} = \begin{pmatrix} \Psi F_1 & \cdots & \Psi F_n \end{pmatrix} = \Psi F.$$

This result provides our first simplification and consider influences $F : (\mathcal{S}, \Phi) \to (\mathcal{T}, \Psi)$, where the underlying domain system (S, ϕ) is fully connected. For the next simplification we can partition (T, ψ) as well into basins $(T_1, \psi_1), ..., (T_m, \psi_m)$. Again we can stratify the matrix for F into blocks representing probabilities of transitioning into different components of (T, ψ): A similar block multiplication argument to that given above implies that each F_i is covariant, for $i = 1, 2, ..., m$. However, it need not be the case that any F_i is stochastic, since a column from F_i is only a portion of the full column in F.

$$F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{pmatrix} \tag{4}$$

As we will show below for attractor systems, and which we have proven for RDDSs more generally, the columns in a given block F_i all sum to the same value and so are uniform. This implies that although each F_i need not be stochastic, it can be made stochastic after scaling by a positive number. In other words, after possible scaling, F is fully determined by stochastic covariant influence $F_i : (\mathcal{S}, \Phi) \to (\mathcal{T}_i, \Psi_i)$ between single basins for each $i = 1, 2, ..., m$. We will leverage this result through the remainder of the discussion.

First we will expand on what the covariance condition for our influences mean in a general context which we will specialize when looking at attractor dynamics. Consider such a covariant influence $F : (\mathcal{S}, \Phi) \to (\mathcal{T}, \Psi)$. Fix a state $s \in S$ in the underlying RDDS as well as an ordering of the states in S so we can write F as a matrix. Let $e_s \in \mathcal{S}$ denote the probability vector with 1 in the position for the state s and 0s elsewhere. Then the covariance condition says that

$$Fe_{\phi(s)} = F\Phi e_s = \Psi F e_s.$$

For any fixed state $t \in T$ in the underlying RDDS of $(\mathcal{T}, \Psi)$, this equality says that

$$p_F(t|\phi(s)) = \sum_{t' \in \psi^{-1}(\{t\})} p_F(t'|s), \tag{5}$$

where $\psi^{-1}(\{t\})$ is the collection of states in T which have t as a direct descendant (i.e. $t' \in \psi^{-1}(\{t\})$ if and only if $\psi(t') = t$). Note that if $\psi^{-1}(\{t\})$ is empty, for example if t is parentless, then the sum on the right hand side of Eq. (5) is 0. This case can be seen as a special case of the ancestry condition from Eq. (1) as it states that the successor of a state s cannot transition to a state with no parents. Running this argument inductively using Eq. (5) we can re-derive the full ancestry condition for stochastic covariant influences.

4.2. Attractor Dynamics

The results in Sec. 4.1 allow us to greatly simplify our next steps by considering covariant influences between systems with one basin of attraction each.

Since we are currently considering cycle-only dynamics, every state t in T has exactly one parent. Thus, if t' is this unique parent of t, so $\psi(t') = t$, then Eq. (5) transforms into

$$p_F(\psi(t')|\phi(s)) = p_F(t'|s). \tag{6}$$

Applying this result inductively allows us to observe that $p_F(t'|s) = p_F(\psi^k(t')|\phi^k(s))$ for any natural number $k \geq 0$. Physically these observations imply that the covariance constraint forces transition probabilities to respect time evolutions for cycle-only dynamics in the sense that probabilities are preserved by evolving both systems simultaneously in time.

The preservation of probabilities under time evolution in Eq. (6) suggests a connection between our covariant influences and the circulant matrices of Sec. 3.2. To this effect, we can expand F as in Eq. (4) and consider the covariant influence $F_i : (\mathcal{S}, \Phi) \to (\mathcal{T}_i, \Psi_i)$. Since (T_i, ψ_i) consists of a single cycle we can order the states in T_i as $t_1, ..., t_m$ such that $\psi(t_j) = t_{j+1}$ for each $j = 1, 2, ..., m$, where the index is computed modulo m so that $\psi(t_m) = t_{m+1} = t_1$. With this ordering fixed, the constraint in Eq. (6) says exactly that as we go down the rows in F_i. the terms cycle to the left, as in Eq. (2).

These observations tell us that the transition blocks in F between individual cycles, F_i, are exactly circulant matrices corresponding to the lengths of the cycles T_i in T. Using the basis description for circulant matrices in Sec. 3.2, we can give an explicit expression for the possible covariant influences F_i. In particular, if we fix any state t in T_i and any state s in S, then after ordering the rest of the states in terms of the dynamics on S and T we can describe F_i by

$$F_i = \sum_{k=0}^{\gcd(m,n)-1} p_F(t|\phi^k(s)) C^{(k)}_{m,n} = \begin{pmatrix} p_F(t|s) & p_F(t|\phi(s)) & \cdots & p_F(t|\phi^{\gcd(m,n)-1}(s)) \\ p_F(t|\phi^{\gcd(m,n)-1}(s)) & p_F(t|s) & \cdots & p_F(t|\phi^{\gcd(m,n)-2}(s)) \\ \vdots & \vdots & \ddots & \vdots \\ p_F(t|\phi(s)) & p_F(t|\phi^2(s)) & \cdots & p_F(t|s) \end{pmatrix}. \tag{7}$$

The formula in Eq. (7) can be hard to interpret at first glance, but has an incredibly simple and powerful consequence. In particular, from the cyclicity of the entries in F_i, it can be seen that each column is a cyclic permutation of the previous column. This immediately shows that the sum of probabilities in any two columns are equal, allowing us to perform the simplifications described at the end of Sec. 4.1.

In addition to this simplification result, looking at the repetition of entries in any given column we can see that the number of free probabilities in the block is exactly $\gcd(m, n)$, and each probability occurs exactly $\frac{m}{\gcd(m,n)}$ times. Since F_i is a block in the stochastic matrix F, the entries in any one of its columns must sum to at most one. Together with the repetition of probabilities this gives a strict bound on our probability $p_F(t|s)$:

$$p_F(t|s) \leq \frac{1}{m/\gcd(m,n)} = \frac{\gcd(m,n)}{m}. \tag{8}$$

To illuminate how we can interpret the bound in Eq. (8), consider the following collection of three disjoint cycles in Fig. 3 with individual covariant influences between them.

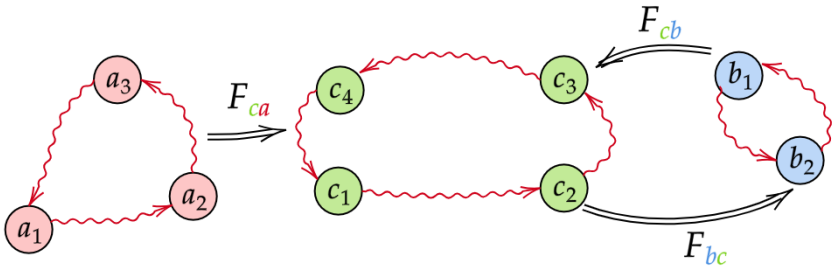

Figure 3. Diagram representing covariant influences between cycles of relatively prime lengths, 2 and 3, and cycles of dividing lengths, 2 and 4.

Using our previous classification, the covariant influences F_{cb} and F_{bc} consist of two free transition probabilities each, given by $p_1 := p(c_1|b_1)$, $p_2 := p(c_1|b_2)$ and $q_1 := p(b_1|c_1)$, $q_2 := p(b_1|c_2)$, respectively. On the other hand, F_{ca} has only one free transition probability,

$p := p(c_1|a_1)$. In general, if F_{ca} is part of a larger stochastic covariant influence, then p is bounded by $1/4$ since the c cycle is of length 4. On the other hand, if F_{ca} is a standalone influence, then the stochasticity condition forces $p = 1/4$. In summary, when dealing with each of these influences as standalone stochastic matrices, we can write

$$F_{cb} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \\ p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}, \quad F_{bc} = \begin{pmatrix} q_1 & q_2 & q_1 & q_2 \\ q_2 & q_1 & q_2 & q_1 \end{pmatrix}, \quad F_{ca} = \begin{pmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{pmatrix},$$

where q_1 and q_2 can take on any value between 0 and 1, as long as $q_1 + q_2 = 1$, while p_1 and p_2 can only be at most $1/2$, as described in Eq. 8.

These computations show that the bound in Eq. (8) provides a kind of stochastic version of the divisibility rule in Eq. (1). Indeed, if we are considering two cyclic discrete dynamical systems, one of length ℓ_1 and the other of length ℓ_2, then if ℓ_2 divides ℓ_1 Eq. (1) guarantees that we have a deterministic covariant influence from the ℓ_1 cycle to the ℓ_2 cycle. In the stochastic setting this implies that we can have any transition probabilities between 0 and 1. But if ℓ_2 does not divide ℓ_1 then although a stochastic covariant influence from the ℓ_1 cycle to the ℓ_2 cycle exists, its probabilities are strictly less than 1. If we go to the extreme where ℓ_1 and ℓ_2 only have 1 as a common divisor, then all transition probabilities are identical, and so they are maximally constrained.

4.3. Transient Dynamics

The next step to understanding covariant influences is to introduce transient states, or in other words branches as seen in blue in Fig. 1. Transient states introduce non-reversibility to our dynamical systems, and hence greatly increase the complexity of studying covariant

influences that can act on them. Going back to the entry-wise covariance condition in Eq (5), observe that $\psi^{-1}(\{t\})$ is exactly the set $P_1(t)$ of parents states of t.

The reason for this change in perspective is the graphical intuition it provides and the ease of generalization. In particular, if $P_k(t)$ for $k \geq 1$ denotes the set of kth order ancestors of t, i.e. all states t' such that $\psi^k(t') = t$, then we can provide a simple description of the formula obtained by applying Eq. (5) inductively. However, to perform this generalization we first must fix a chain of states $s, s_1, ..., s_k$ such that $\phi^i(s_k) = s_{k-i}$ for $i = 0, 1, ..., k$, where $s_0 := s$. This chain can be thought of as a fixed k-step history of the state s. With this chain and notation fixed, the general form of Eq. (5) becomes

$$p_F(t|s) = \sum_{t' \in P_k(t)} p_F(t'|s_k). \tag{9}$$

This classification provides an elegant way of proving the ancestry and transient progeny constraints in Eq. (1) for the stochastic setting. Indeed, for ancestry, if $t \in T$ and $s \in S$ are states such that $a_0(\psi^k(t)) < a_0(\phi^k(s))$, then we have a $a_0(\psi^k(t)) + 1$-step time history of $\phi^k(s)$: $\phi^k(s), s_1, ..., s_{a_0(\psi^k(t))+1}$. With this time history we can use Eq. (9) to write

$$p_F(\psi^k(t)|\phi^k(s)) = \sum_{t' \in P_{a_0(\psi^k(t))+1}(\psi^k(t))} p_F(t'|s_{a_0(\psi^k(t))+1}).$$

But $a_0(\psi^k(t))$ is the distance to the farthest ancestor of $\psi^k(t)$, so $P_{a_0(\psi^k(t))+1}(\psi^k(t))$ must be empty, implying that the sum and hence $p_F(\psi^k(t)|\phi^k(s))$ is zero. Since we can use Eq. 9 to expand $p_F(\psi^k(t)|\phi^k(s))$ in terms of k-th ancestors, which includes $p_F(t|s)$, the fact that $p_F(\psi^k(t)|\phi^k(s)) = 0$ implies that we must also have $p_F(t|s) = 0$. The transient progeny constraint is then a simple consequence of the ancestry one since $d(s) \geq d(t)$ is equivalent to requiring that $\phi^{d(s)}(t)$ is an attractor state, or in other words $a_0(\phi^{d(s)}(t)) = +\infty$. But $\phi^{d(s)}(s)$ is always an attractor state by definition of the transient

progeny, so $a_0(\phi^{d(s)}(t)) = +\infty$ is equivalent to $a_0(\phi^{d(s)}(s)) \leq a_0(\phi^{d(s)}(t))$.

The main contrast between the deterministic and stochastic settings is that these two constraints alone are actually not only necessary, but also sufficient for the existence of a stochastic covariant influence that has a non-zero probability of transitioning between two states. The full proof of this claim is lengthy, but we can give a sketch of the argument here. The basic idea is to prove the claim inductively, where the induction is performed on the size of the transient branches in a dynamical system. For the first step of the proof we use the fact that we can always have a non-zero transition probability into states that live in cycles. This observation is obtained by iterating the dynamics a sufficient number of times so that all states in the domain lie in a cycle, and then using the theory for cycle-only systems developed using circulant matrices in Sec. 4.2.

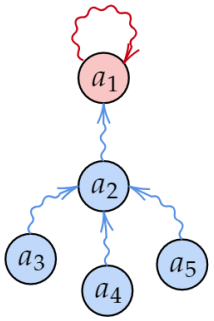

Figure 4. Diagram showing chain with a bouquet of parentless states on the end.

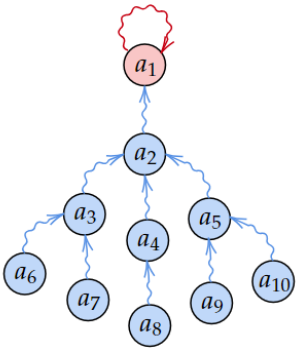

Figure 5. Transient tree obtained by adding bouquets to the parentless states in Fig. 4.

After this initial step, we show that we can always reach a state in a transient chain out of a parentless state. With this step in hand the next is a bouquet procedure, depicted in Fig. 4, where we flower out the parentless state at the end of a chain that otherwise has no branches. Finally, to complete the argument we need to be able to consider a transient tree with an arbitrary number branches. The key observation is that such a tree can be built inductively by adding bouquets on parentless states. For example, if we add bouquets on the parentless states in Fig. 4 we can construct the tree seen in Fig. 5. This

approach allows us to construct our desired covariant influence.

5. Conclusion

Throughout this thesis we have constructed a general treatment for working with random discrete dynamical systems (RDDS) subject to external influences that act on extended time scales. We introduced and leveraged the resource theory of RDDSs developed by Scandolo et al. in [1] to encode long time scale external influences mathematically as covariant influences, and determine what structure they take. In this analysis we first used the theory of resources as a tool for constructing dynamical systems. In the process we demonstrated that an arbitrary stochastic covariant influence can be partitioned into covariant influences between individual basins of attraction. With this foundation developed, we were able to provide a complete description of covariant influences between systems without transient states. Unlike in the deterministic case in [1], where a certain divisibility condition was required to ensure the existence of covariant influences between different systems, we showed that stochastic covariant influences always exist between such systems. However, we also showed that transition probabilities are constrained when the divisibility rule from the deterministic setting in [1] is violated. Finally, we added transience and showed that the theory of stochastic influences gives an elegant form to the necessary and sufficient conditions for having probabilistic transitions between states. Explicitly, using an inductive argument on the structure of transient branches in our dynamical graphs, we were able to show that the ancestry and transient progeny constraints in Eq. (1) are sufficient for the existence of a non-zero transition probability between two states.

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