Resource Theories For Random Discrete Dynamical Systems

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Gene Regulation

Dynamical Model: [Bornholdt et al.]

- Cell behaviour by gene expression and repression
- Gene regulation conducted by protein interactions
- Random Boolean networks reproduce experimental dynamical trajectories

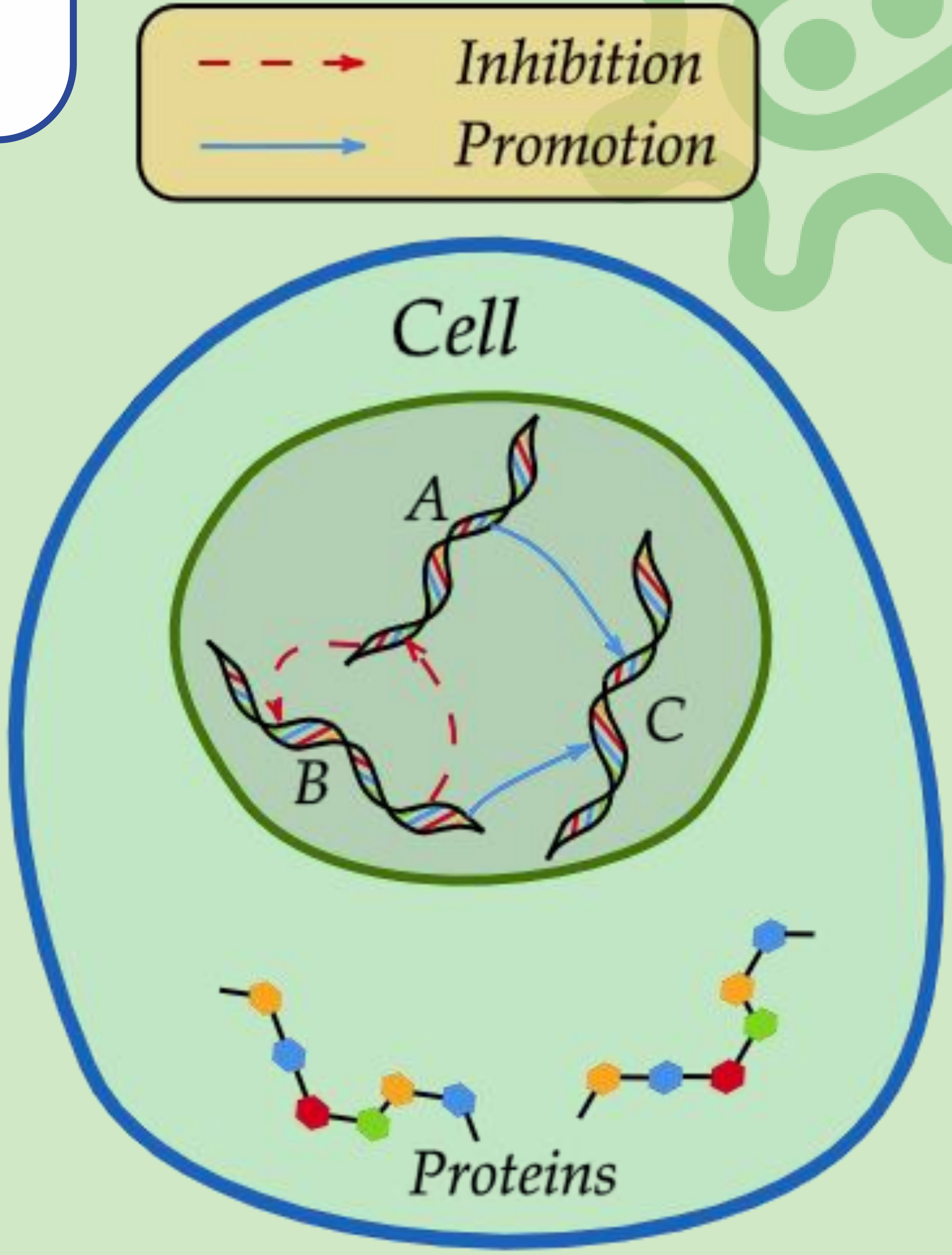

Fig 1. Model of gene regulation.

Gene Regulation

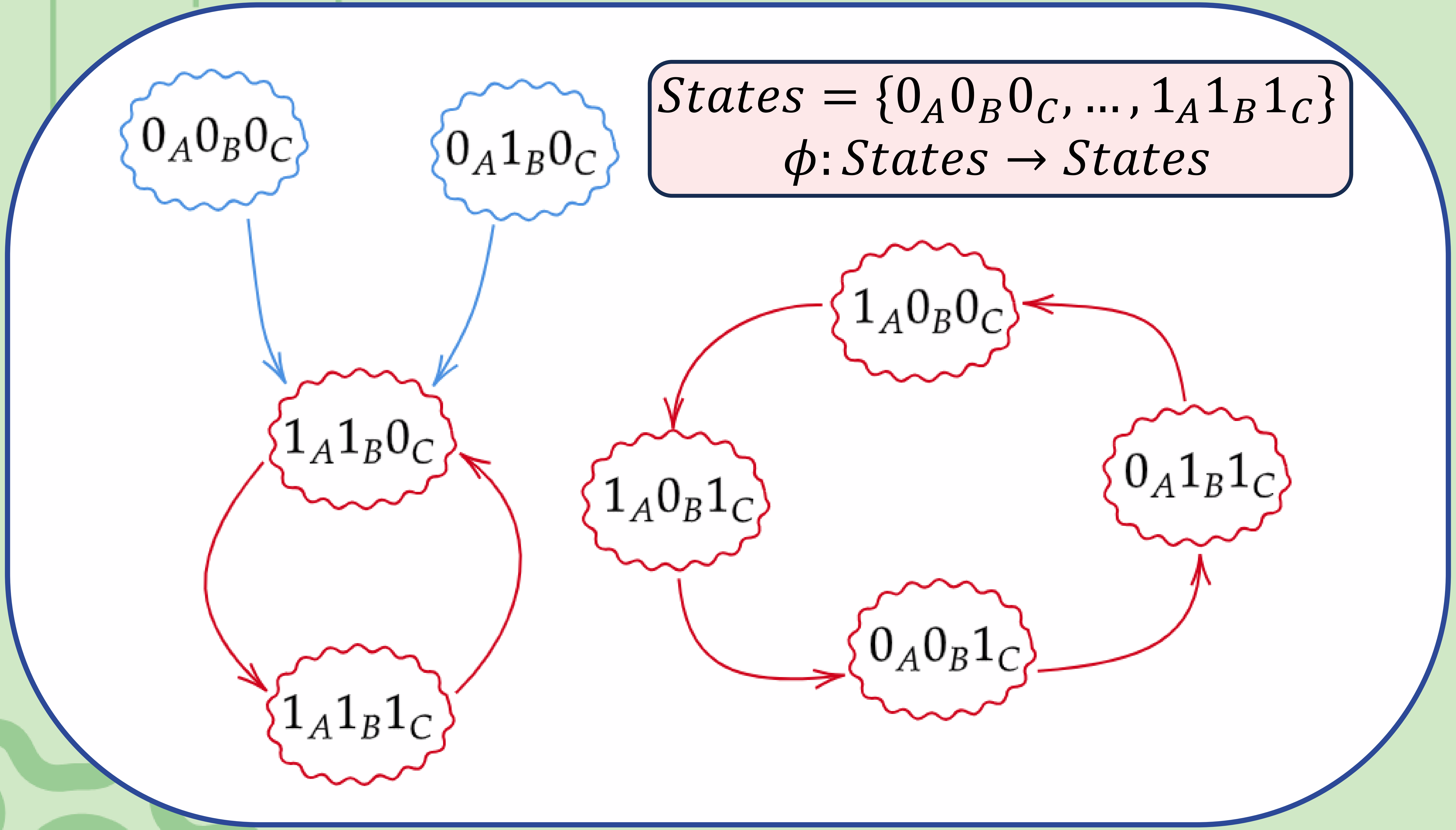

Fig 2. Discrete dynamical system describing gene regulation [Bornholdt et al.].

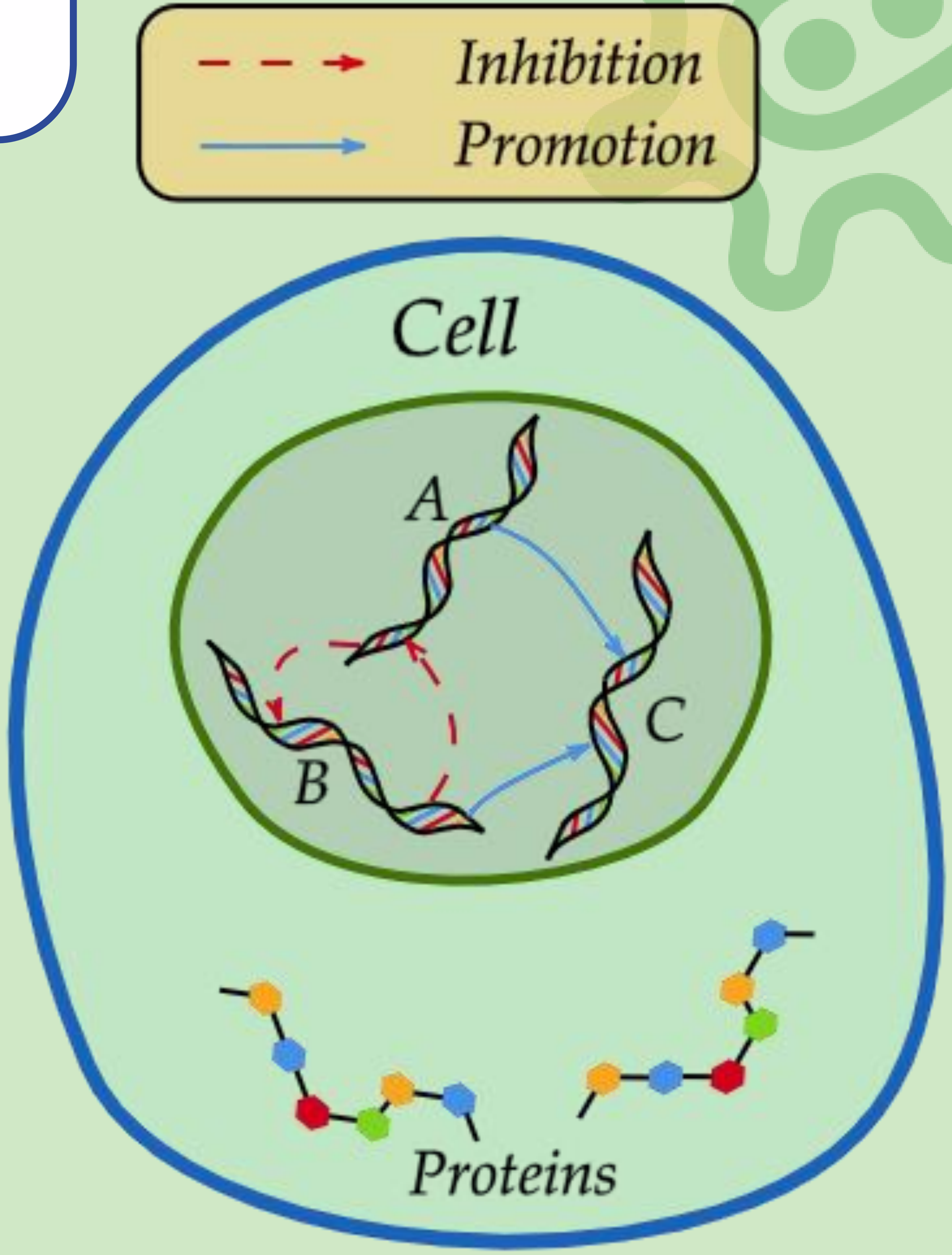

Fig 1. Model of gene regulation.

Historical

System Dependent

Solutions heavily depend on specific system

Perturbation Paradigm

External effects are assumed to be small

Goals

Universal

Can be applied independently of system dynamics

External influences

Allow influences which are not small but act over long timescales

Universality: Resource Theories

Resource Theories:

A resource theory $\mathfrak{R}$ consists of:

- ❑ Resources, A, B, C, …
- ❑ For resources A, B, a restricted class of processes, $\mathfrak{R}(\mathrm{A} \to B)$, converting A to B at no cost.

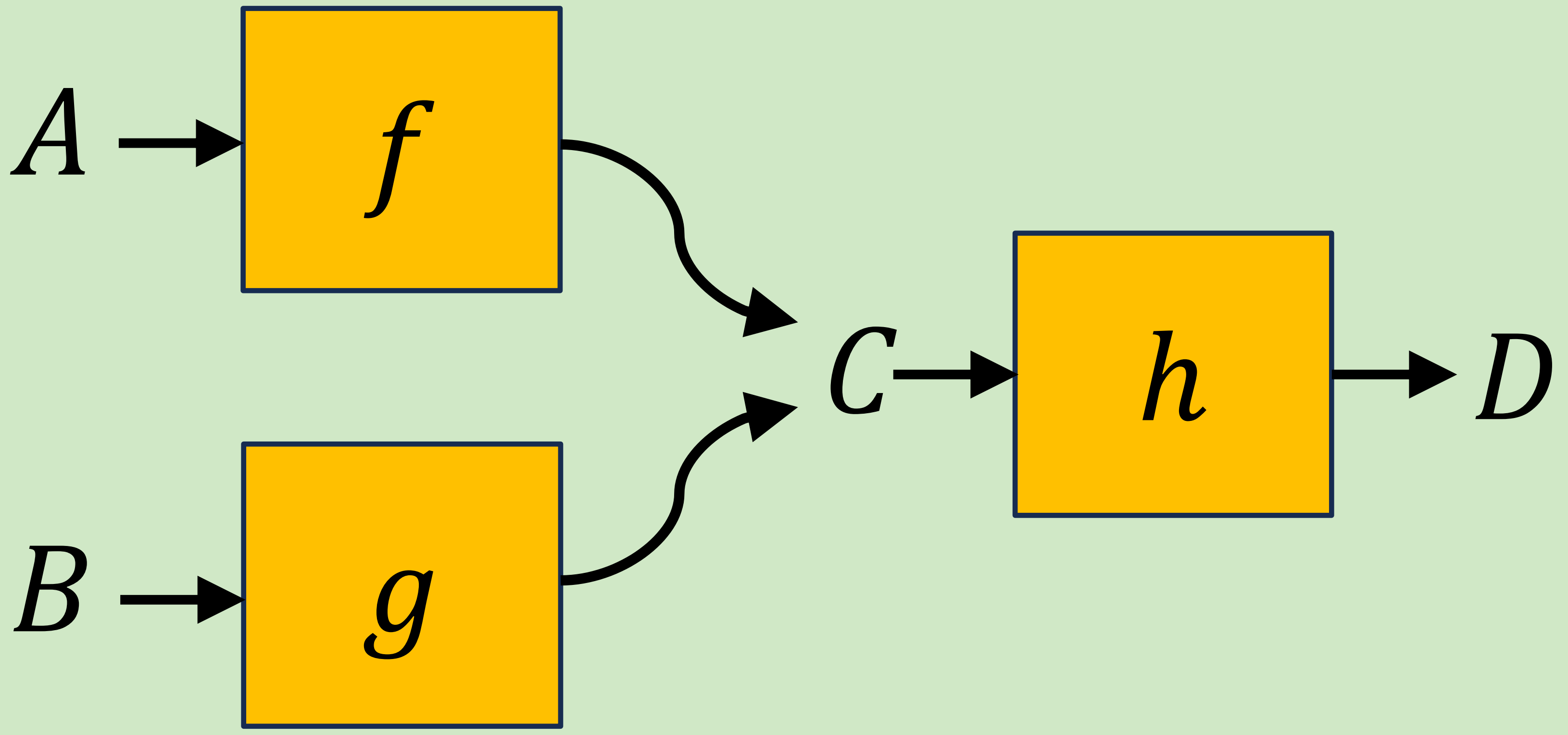

Fig 3. Resources and free processes in a resource theory [Coecke et al.].

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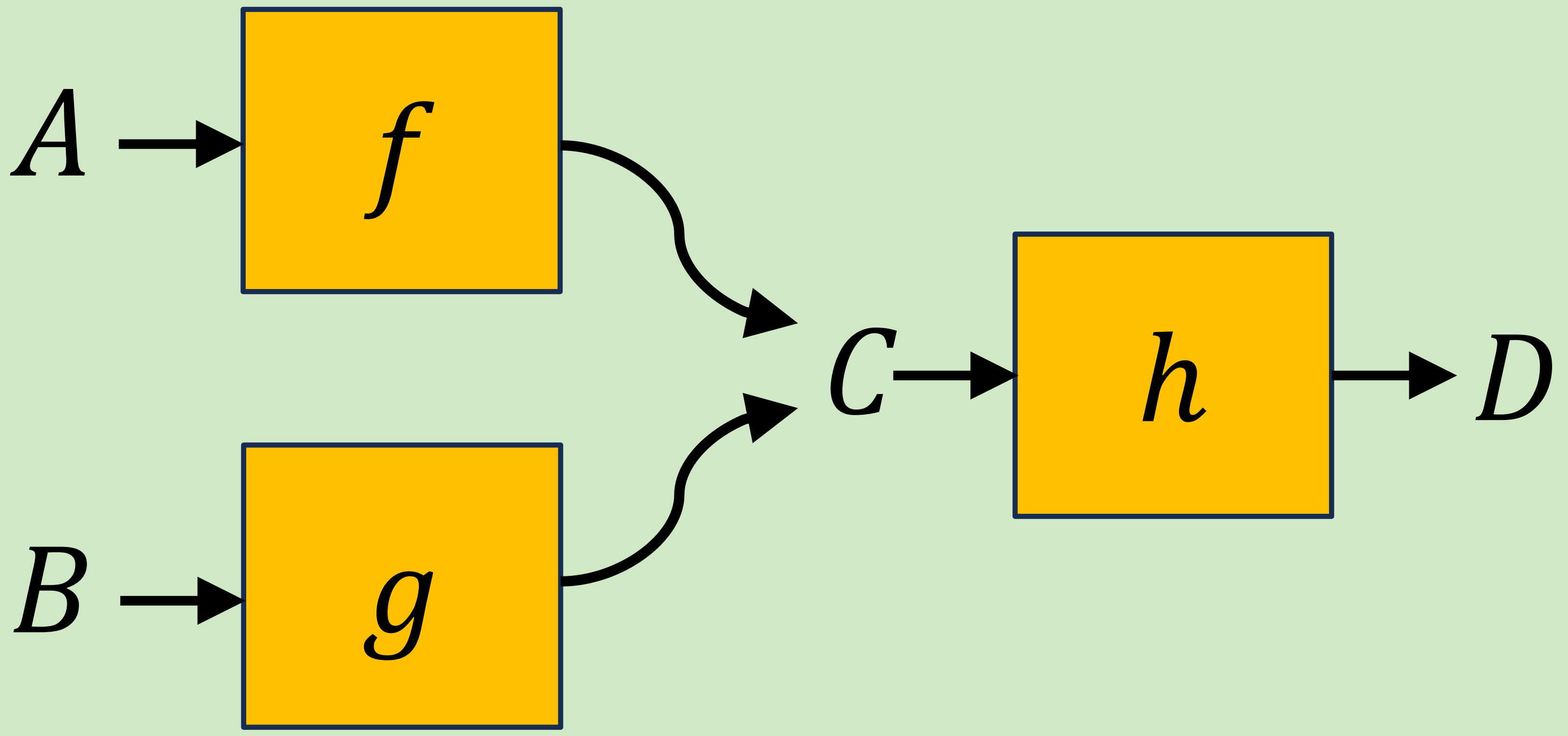
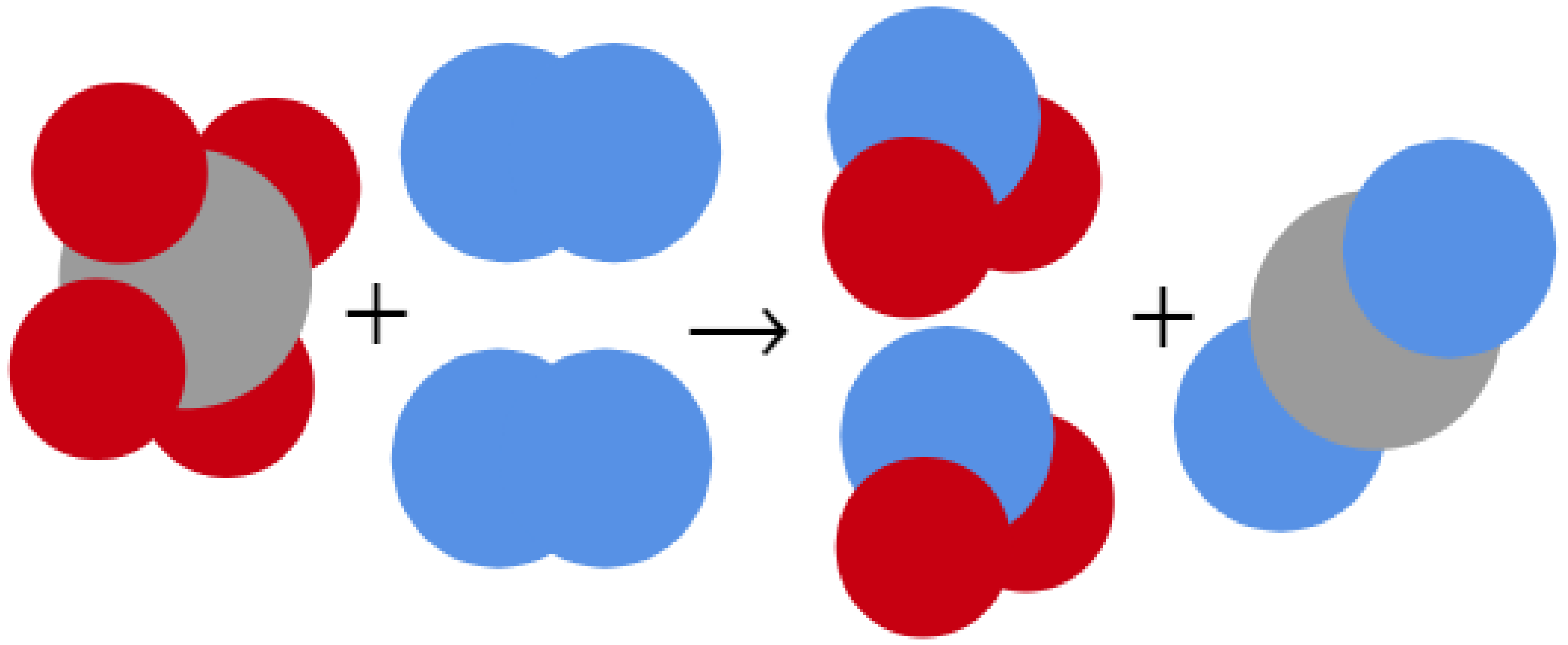


Fig 3. Resources and free processes in a resource theory [Coecke et al.].

Chemical Reactions:



$$CH_4 + 2O_2 \to 2H_2O + CO_2$$

Fig 4. Chemical reaction of oxygen and methane creating water and carbon dioxide [Fritz].

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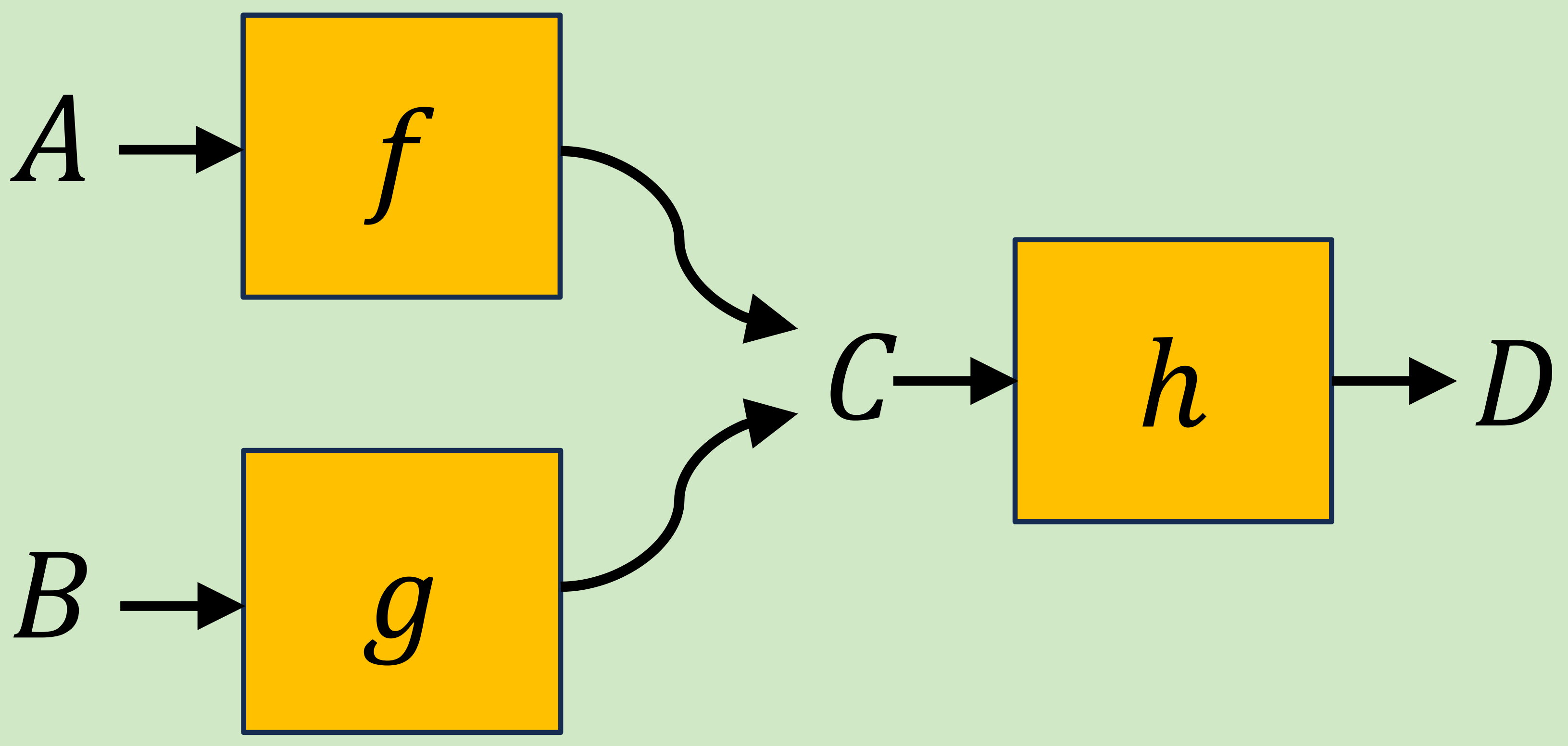


Fig 3. Resources and free processes in a resource theory [Coecke et al.].

Dictionary:

- Resources = states of dynamical systems
- Free processes = influences which act over extended time scales [Scandolo et al.]:

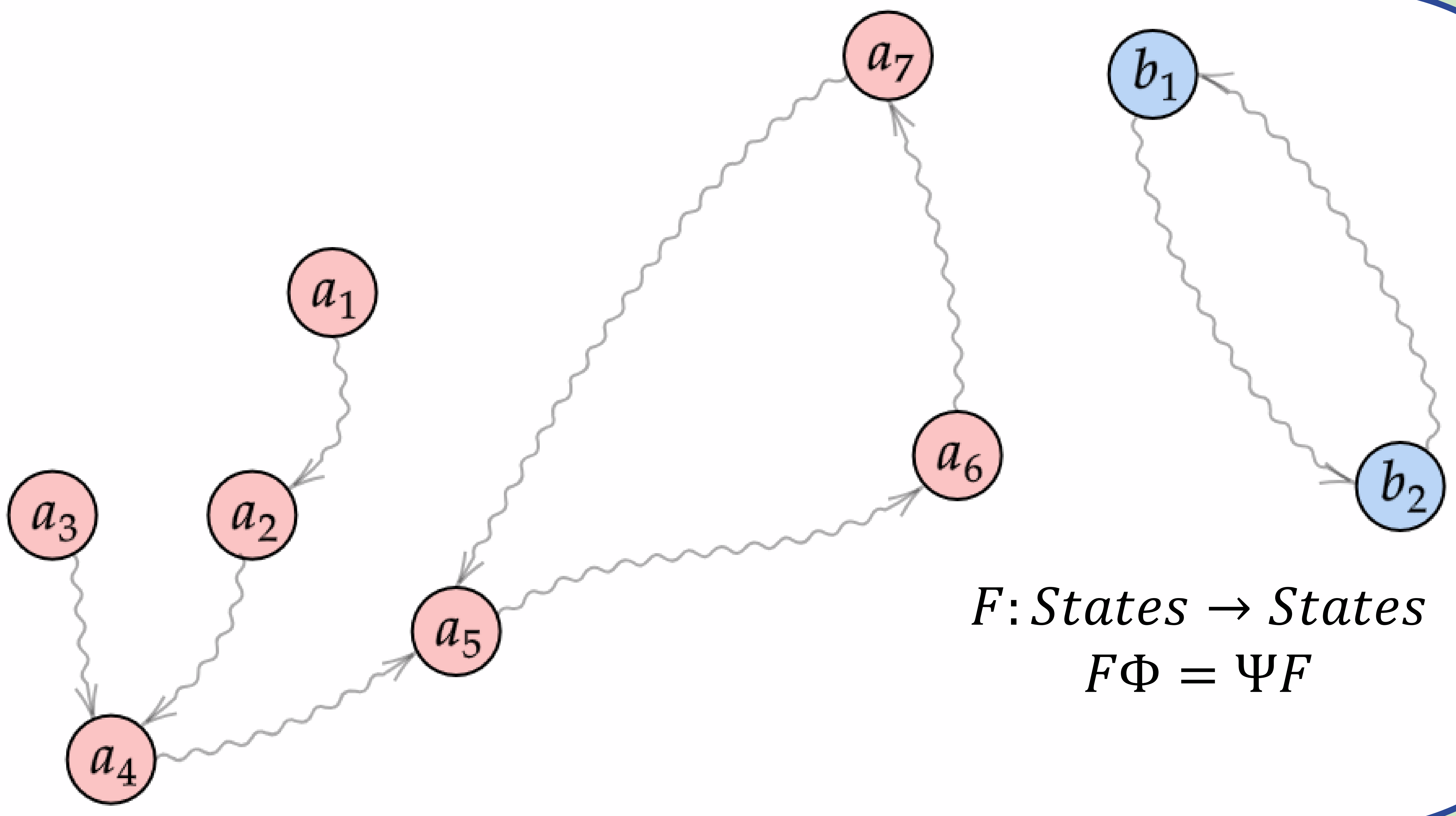
$$F: (\mathcal{S}, \Phi) \to (\mathcal{T}, \Psi)$$
$$F\Phi = \Psi F$$

Q. For states s and t, when can we find an F with $F(s) = t$?

Deterministic Setting: [Scandolo et al.]

Transitions require:

- ❑ Reduction of transience between states
- ❑ Cycle length divided between states



$$F\colon States \to States$$
$$F\Phi = \Psi F$$

Fig 5. Transitions allowed by deterministic covariant influences.

Deterministic Setting: [Scandolo et al.]

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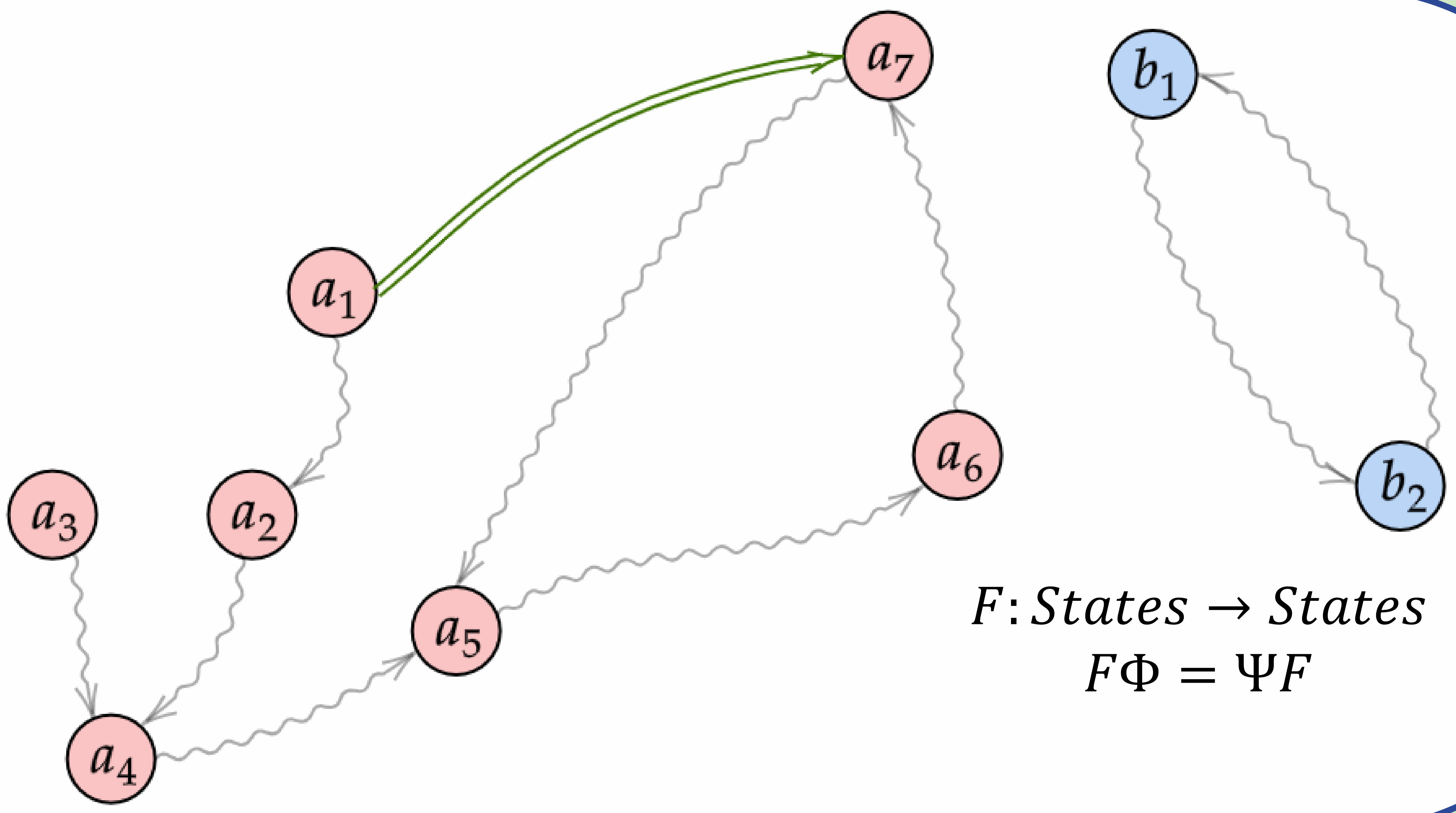

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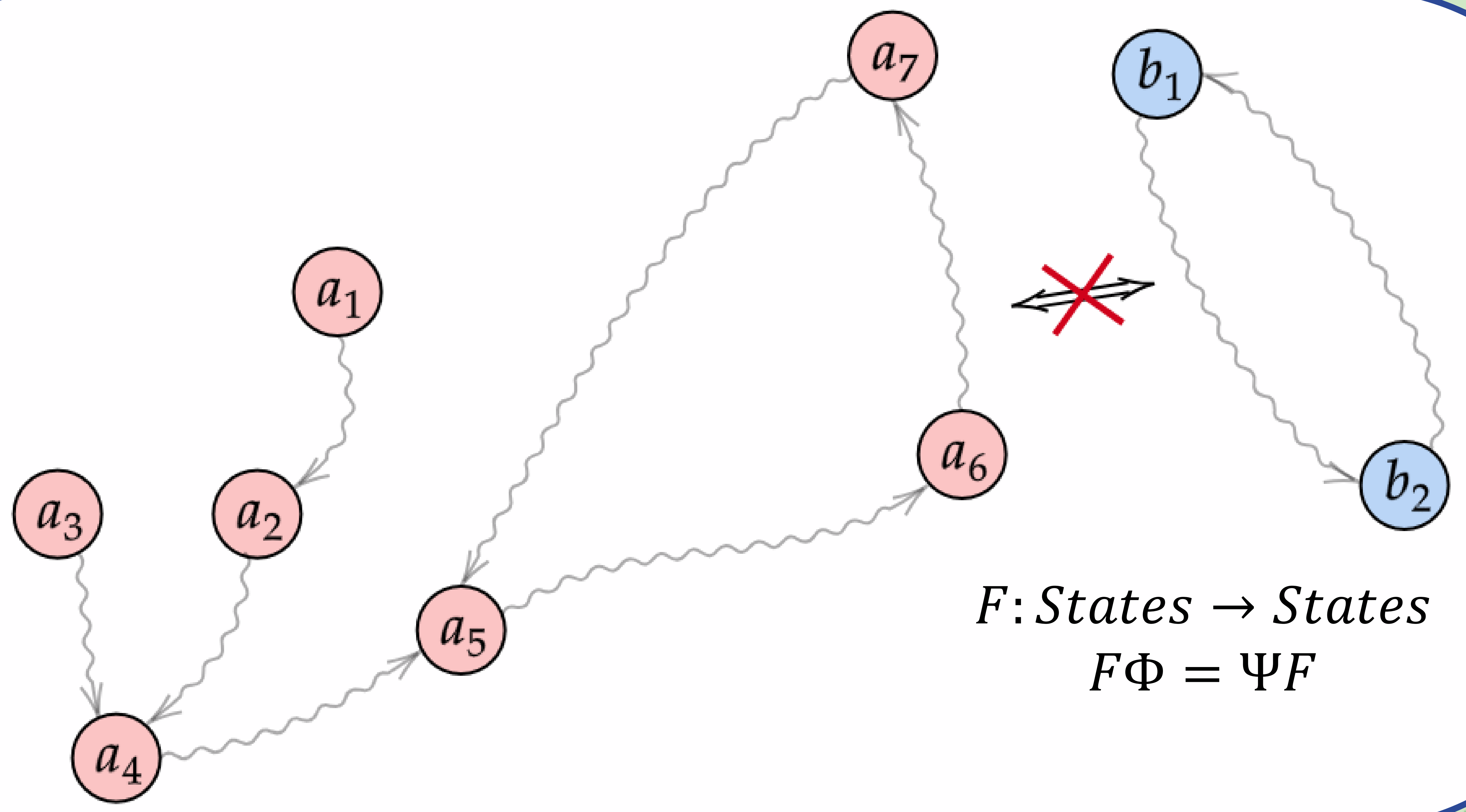

$$F: States \to States$$
$$F\Phi = \Psi F$$

Fig 5. Transitions allowed by deterministic covariant influences.

Stochastic Setting

Influence Matrix

- Functions, $F: States \rightarrow States$, are replaced by stochastic matrices describing state transition probabilities

$$F_{cb} = \begin{pmatrix} p(c_1|b_1) & p(c_1|b_2) \\ p(c_2|b_1) & p(c_2|b_2) \\ p(c_3|b_1) & p(c_3|b_2) \\ p(c_4|b_1) & p(c_4|b_2) \end{pmatrix}$$

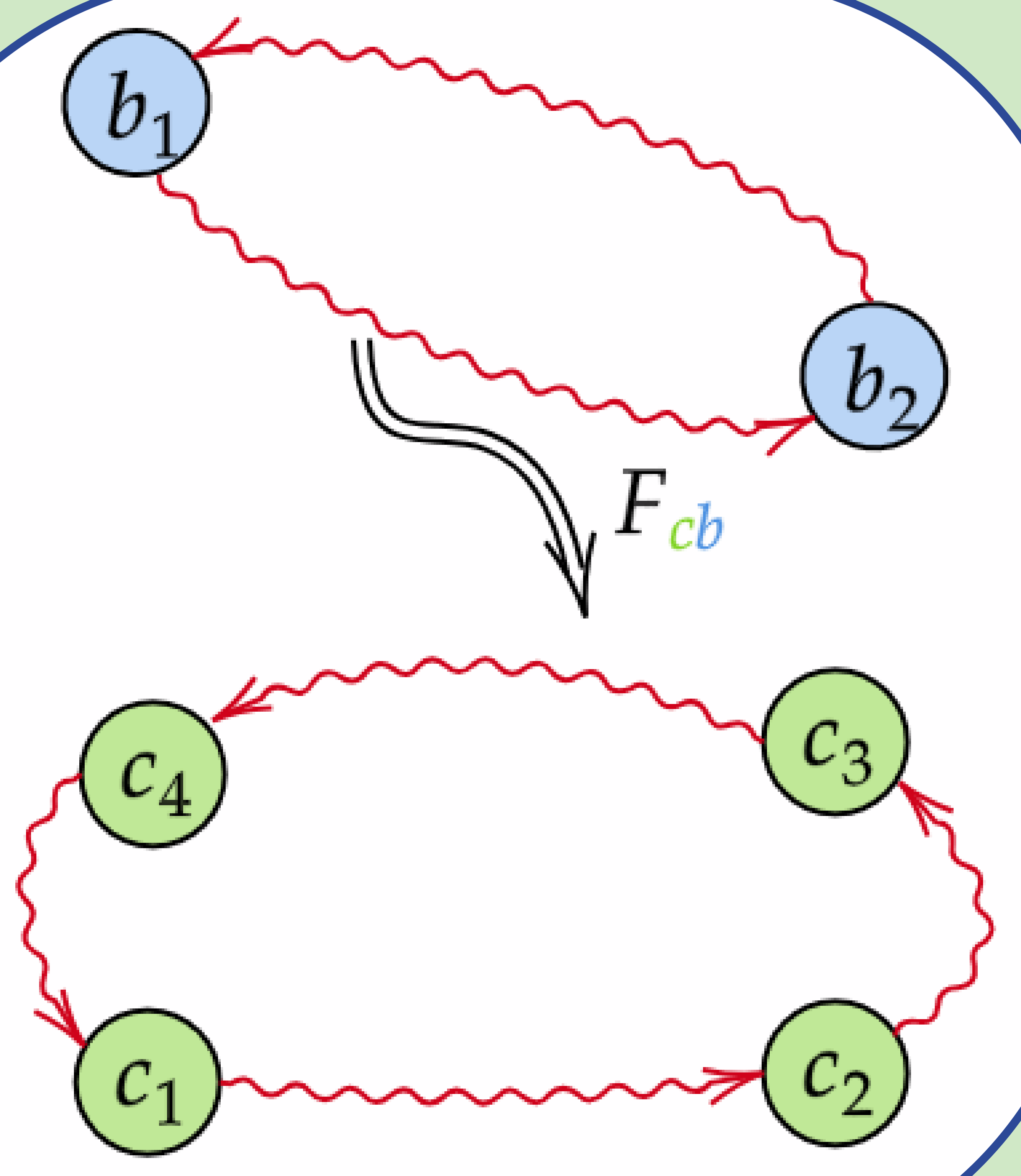

Fig 6. Stochastic transition between two cycles of different lengths.

Circulant Blocks

Influence Matrix

$$F_{cb} = \begin{pmatrix} p(c_1|b_1) & p(c_1|b_2) \\ p(c_2|b_1) & p(c_2|b_2) \\ p(c_3|b_1) & p(c_3|b_2) \\ p(c_4|b_1) & p(c_4|b_2) \end{pmatrix}$$

$\boldsymbol{F\Phi = \Psi F}$

Circulant block

$p_1 := p(c_1|b_1), p_2 := p(c_1|b_2)$

$$F_{cb} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \\ p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}$$

Transitions induced by randomness

Influence Matrix

- Stochastic influences introduce new transitions but with constrained probabilities

$$p(c_1|b_1) := \frac{1}{2}, p(c_1|b_2) := 0$$

$$F_{cb} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \\ 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

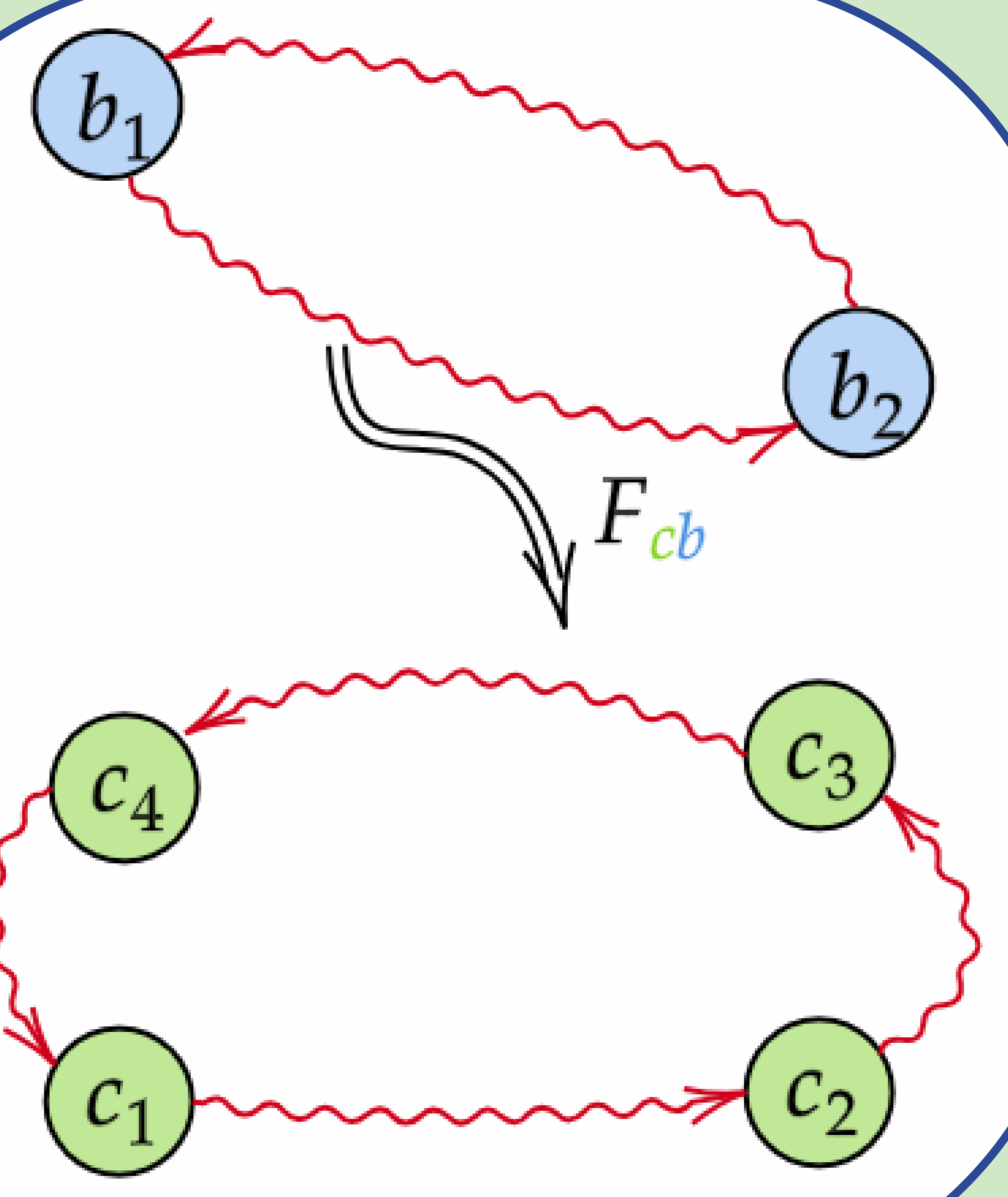

Fig 6. Stochastic transition between two cycles of different lengths.

Bounds on Transition Probabilities

$$F_{cb} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \\ p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}$$

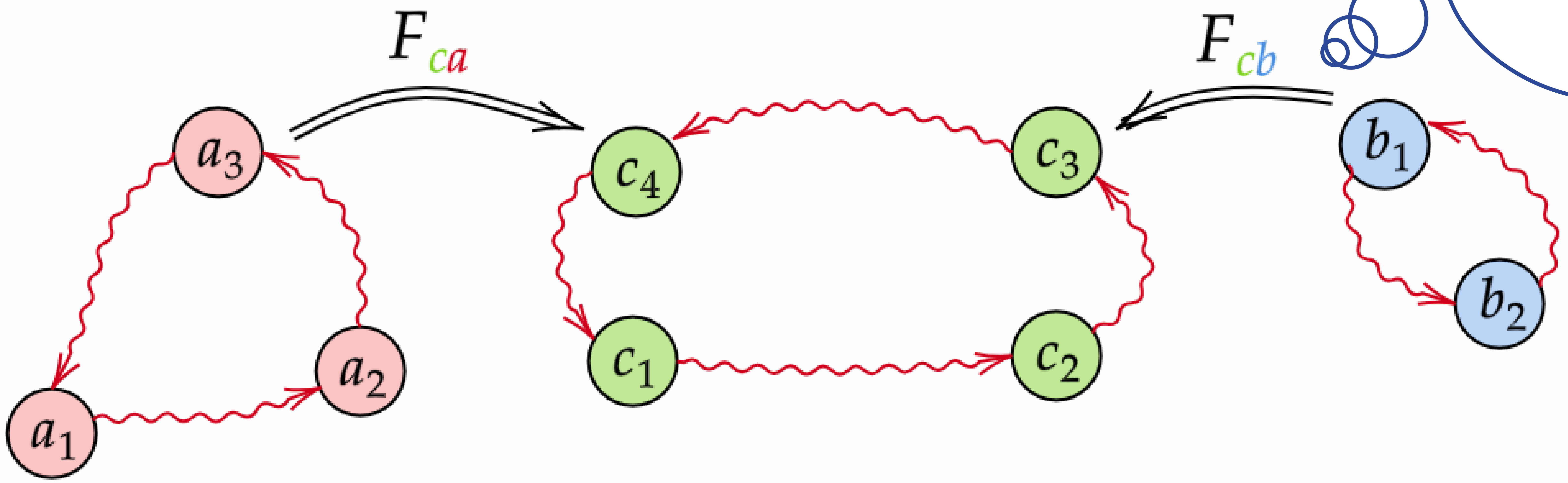

Fig 7. Transitions between cyclic dynamical systems.

Bounds on Transition Probabilities

$$F_{ca} = \begin{pmatrix} p & p & p \\ p & p & p \\ p & p & p \\ p & p & p \end{pmatrix}$$

$$F_{cb} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \\ p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}$$

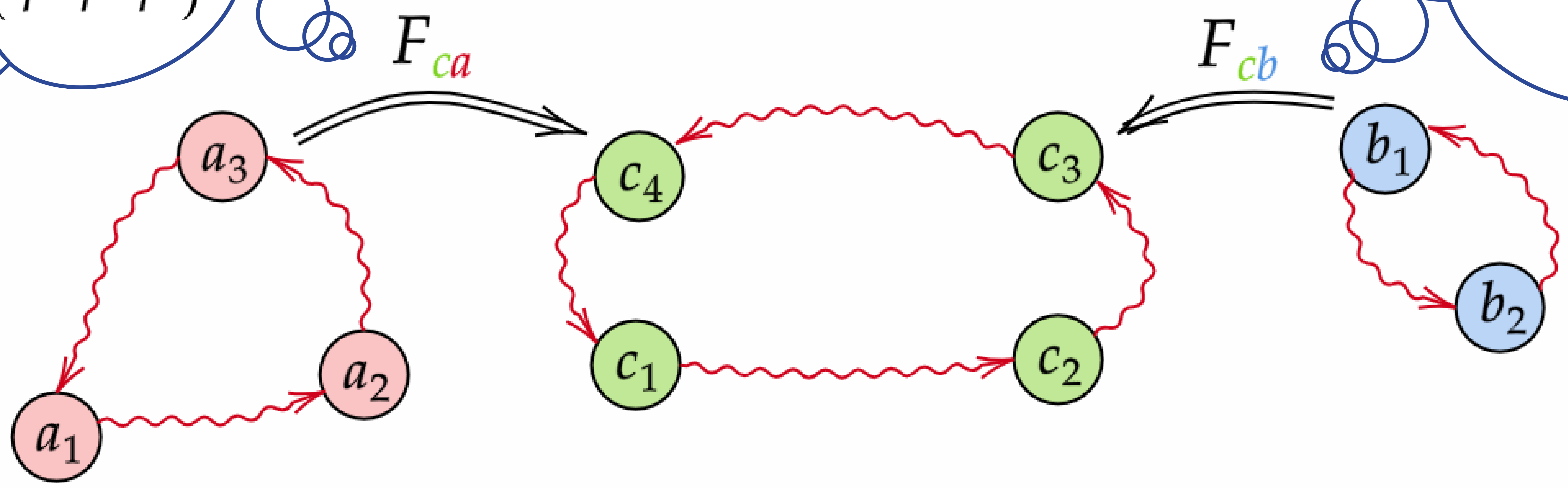

Fig 7. Transitions between cyclic dynamical systems.

Bounds on Transition Probabilities

$$F_{ca} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$p_1 \leq \frac{1}{2} \; and \; p_2 \leq \frac{1}{2}$$

$$F_{cb} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \\ p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}$$

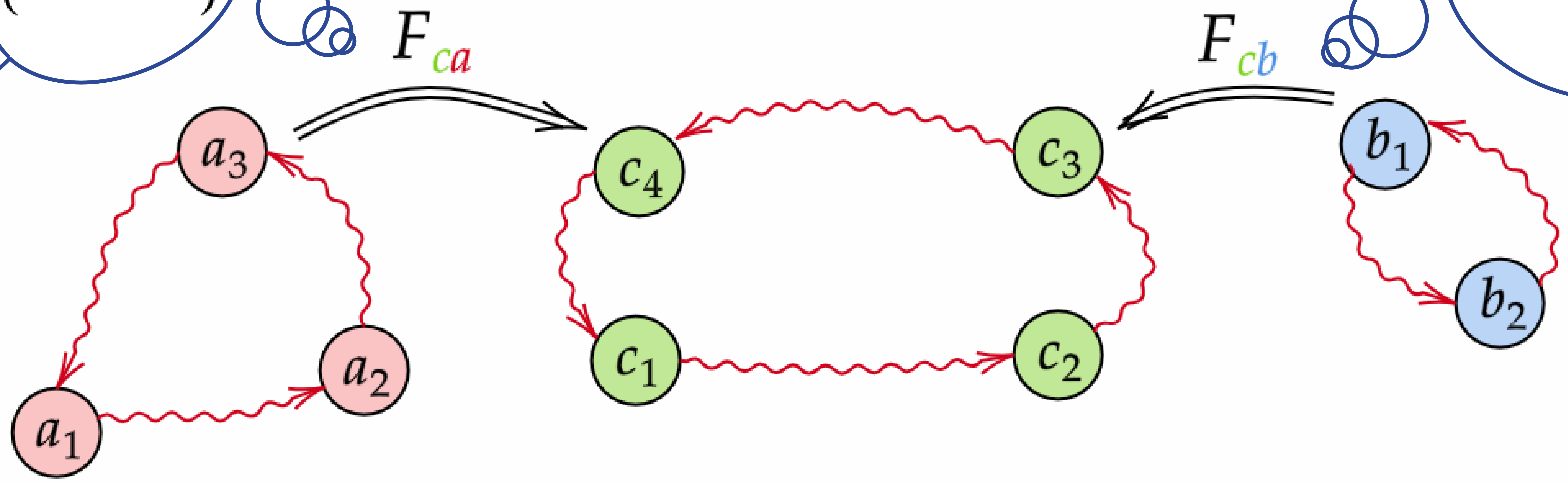

Fig 7. Transitions between cyclic dynamical systems.

Key-Takeaways

01

RDDS' are everywhere

Gene regulatory networks,
Complex systems, chaos theory, etc.

02

Universality

Results apply universally to dynamical systems with different structures

03

Transitions induced from randomness

Random fluctuations induce transitions that are not deterministically allowed

04

Divisibility and probability constraints

Failure of divisibility indicates constraint on transition probabilities

References

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