PRO-REPRESENTABLE VIRTUAL DOUBLE CATEGORIES

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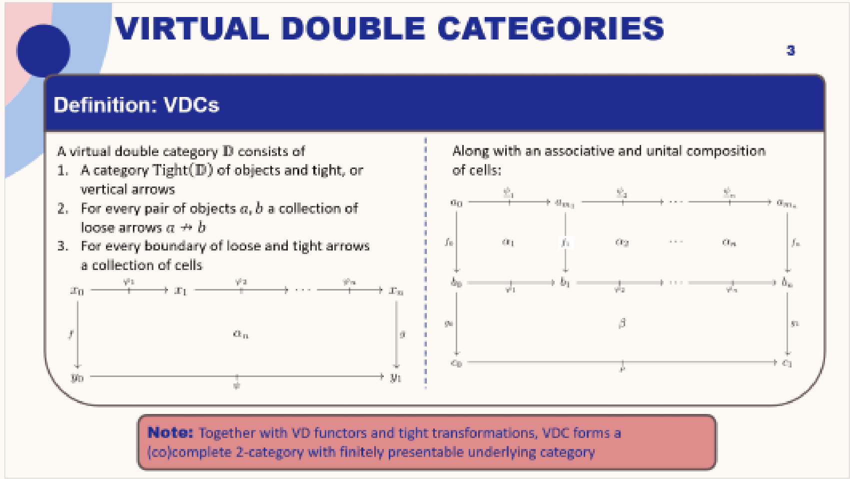
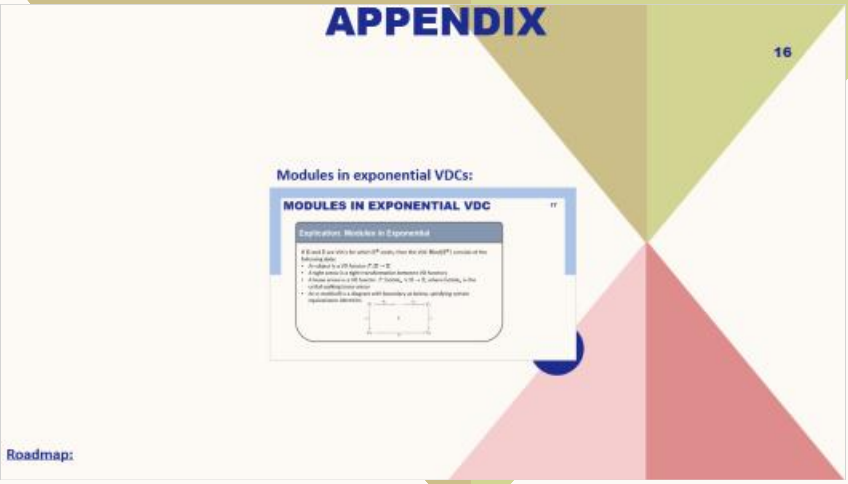
ROADMAP

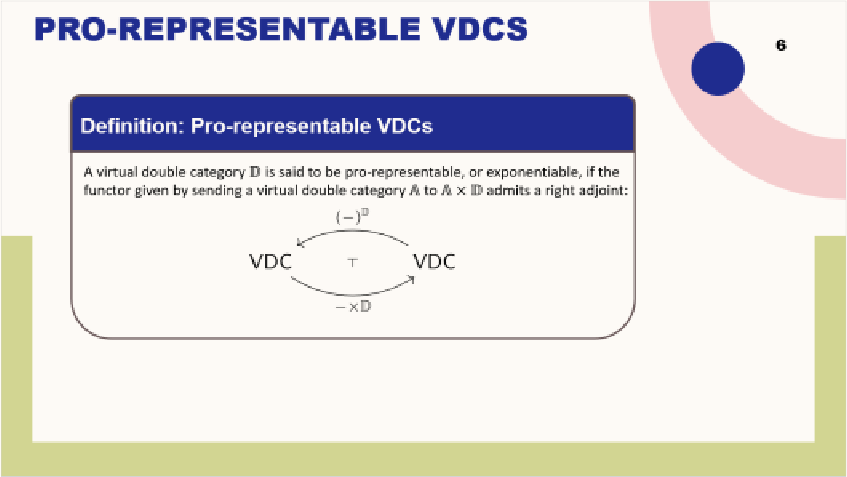
Intro to VDCs:



Appendices:



Pro-representability:



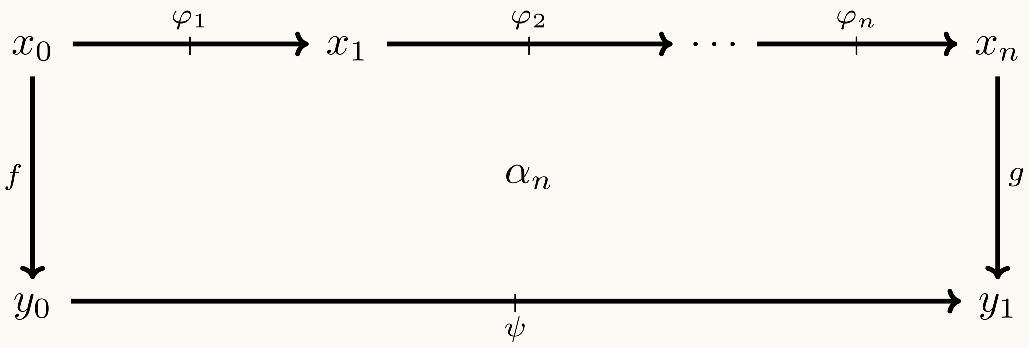
Conclusions and Future Work:

VIRTUAL DOUBLE CATEGORIES

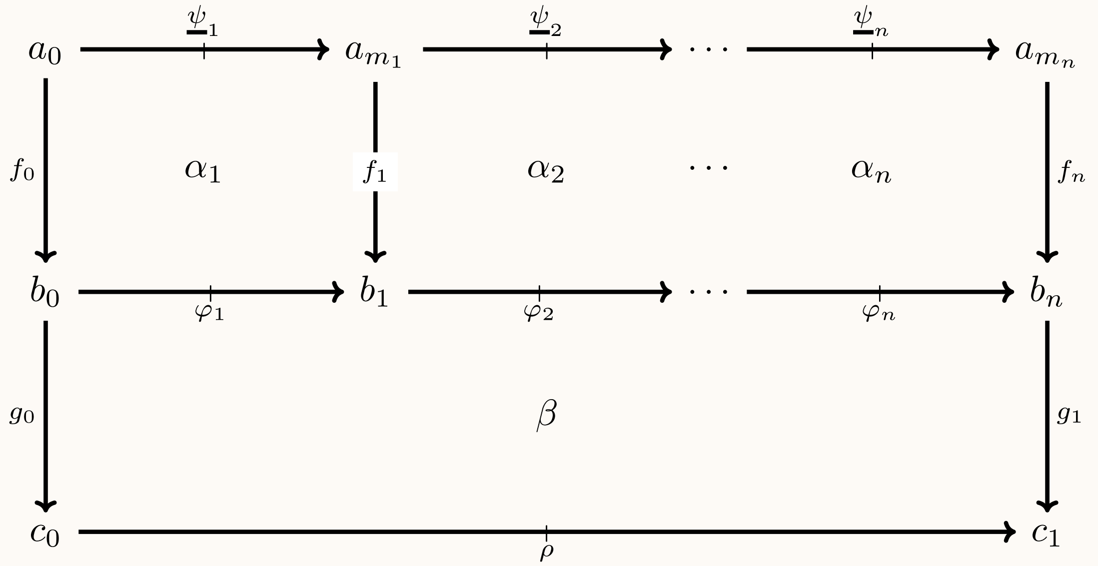
Definition: VDCs

A virtual double category $\mathbb{D}$ consists of

1. A category $\mathrm{Tight}(\mathbb{D})$ of objects and tight, or vertical arrows
2. For every pair of objects a, b a collection of loose arrows $a \nrightarrow b$
3. For every boundary of loose and tight arrows a collection of cells



Along with an associative and unital composition of cells:



Note: Together with VD functors and tight transformations, VDC forms a (co)complete 2-category with finitely presentable underlying category

PSEUDO VS. VIRTUAL CONSTRUCTIONS

(Pseudo-)Double Categories:

- If $\mathcal{E}$ is a category with pushouts, we have a double category $\mathbb{C}\mathrm{ospan}(\mathcal{E})$
- If $(\mathcal{V}, \otimes, I)$ is a monoidal category with finite coproducts that are preserved by $\otimes$, then we have a double category $\mathcal{V}\mathbb{M}\mathrm{at}$
- If $\mathbb{D}$ is a double category with certain reflexive co-equalizers, we have a double category $\mathbb{M}\mathrm{od}(\mathbb{D})$ of monoids in $\mathbb{D}$

Virtual Double Categories:

- For any category $\mathcal{E}$, we have a virtual double category $\mathbb{C}\mathrm{ospan}(\mathcal{E})$
- For any virtual double category $\mathbb{D}$ we have a virtual double category $\mathbb{D}\mathbb{M}\mathrm{at}$
- For any virtual double category $\mathbb{D}$, we have a unital virtual double category $\mathbb{M}\mathrm{od}(\mathbb{D})$ of modules in $\mathbb{D}$

UNIVERSALITY OF VIRTUAL CONSTRUCTIONS

Universality of Constructions on Virtual Double Categories:

- For any category $\mathcal{E}$, the virtual double category $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is the free virtual equipment on $\mathcal{E}$ [DPP10]
- For any virtual double category $\mathbb{D}$ the virtual double category $\mathbb{D}\mathbb{M}\mathrm{at}$ is the free coproduct completion of $\mathbb{D}$ [Kaw25]
- For any virtual double category $\mathbb{D}$, $\mathbb{M}\mathrm{od}(\mathbb{D})$ is the cofree normal completion of $\mathbb{D}$ [CS10]
- For any virtual double category $\mathbb{D}$, $\mathbb{D}\mathbb{P}\mathrm{rof} = \mathbb{M}\mathrm{od}(\mathbb{D}\mathbb{M}\mathrm{at})$ is the free collage cocompletion of $\mathbb{D}$ [Kaw25]

PRO-REPRESENTABLE VDCS

Definition: Pro-representable VDCs

A virtual double category $\mathbb{D}$ is said to be pro-representable, or exponentiable, if the functor given by sending a virtual double category $\mathbb{A}$ to $\mathbb{A} \times \mathbb{D}$ admits a right adjoint:

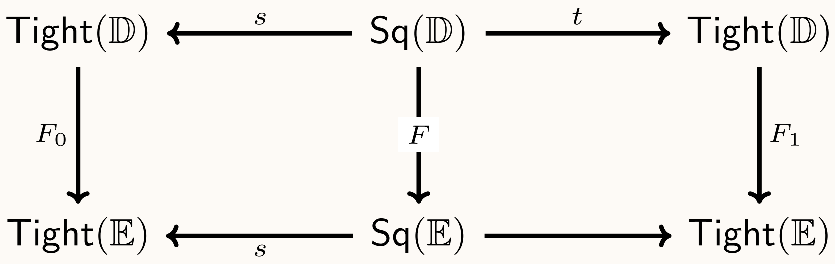
$$\mathsf{VDC} \underset{- \times \mathbb{D}}{\overset{(-)^{\mathbb{D}}}{\underset{\top}{\rightleftarrows}}} \mathsf{VDC}$$

EXPONENTIALS

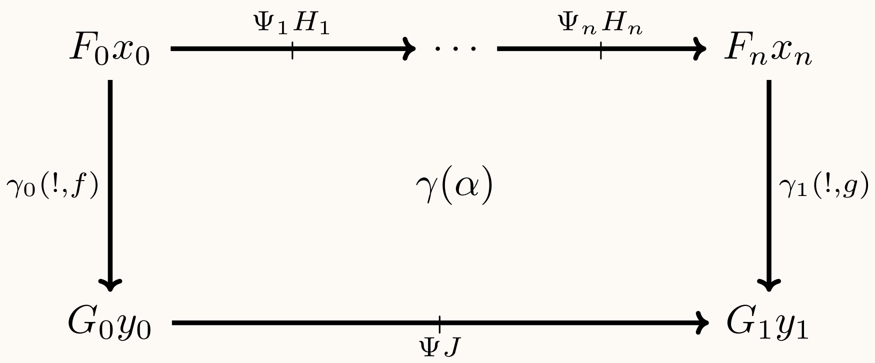
Explication: Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then it must consist of the following data:

- Objects are functors Tight($\mathbb{D}$) →Tight($\mathbb{E}$)
- Tight arrows are natural transformations
- Loose arrows maps of spans



- n-Multi-cells assign to each n-multicell in $\mathbb{D}$ an n-multicell in $\mathbb{E}$ with the following boundary:



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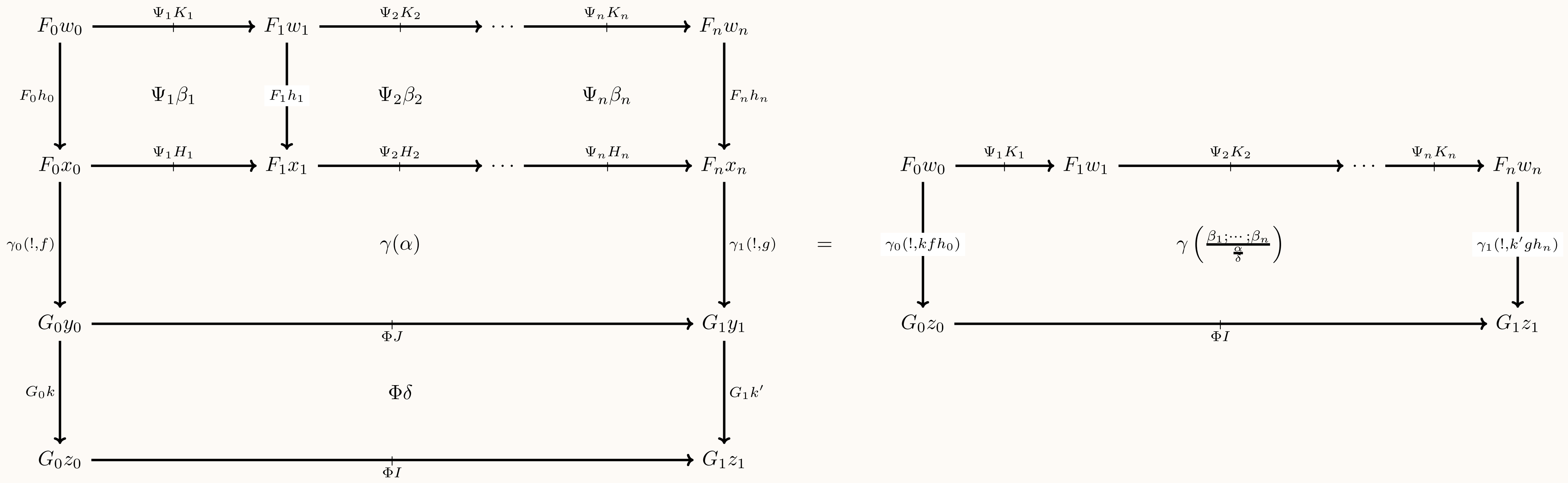
EXPONENTIALS

Explication: Exponential (cont.)

Where multicells are subject to functoriality with respect to vertical pasting:



PRO-REPRESENTABLE VDCS

Theorem: Characterization of Pro-representable VDCs

Let $\mathbb{D}$ be a VDC, and write $\mathbb{D}(\varphi_1, \dots, \varphi_n; \psi) =: \mathbb{D}(\underline{\varphi}; \psi)$ for the set of cells with loose source the sequence $\varphi_1, \dots, \varphi_n$ and with loose target ψ. Vertical pasting can be encoded by functions

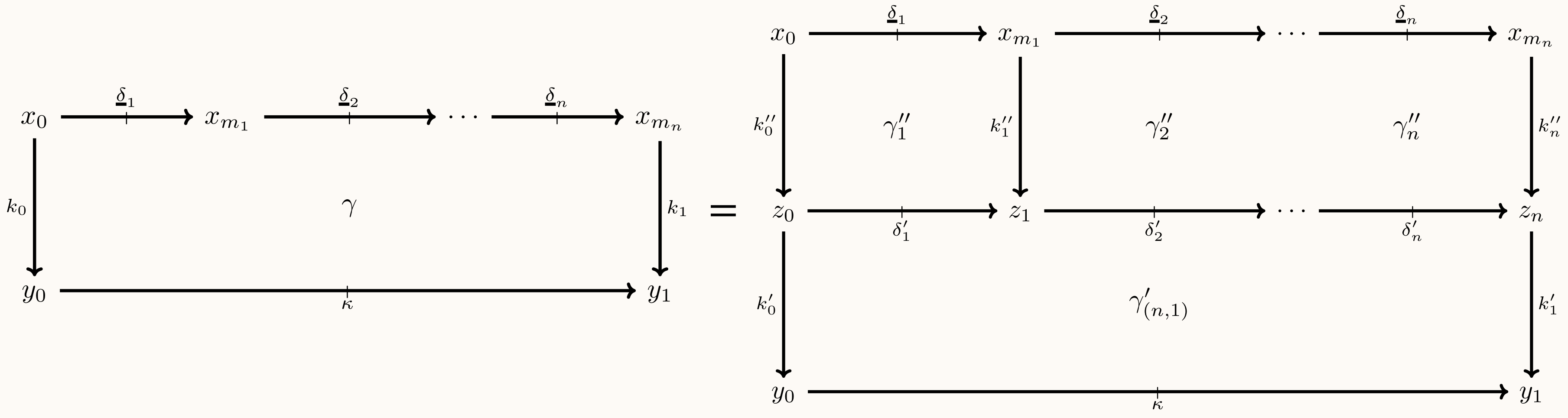
$$\int^{\varphi_i : \mathbb{D}} \mathbb{D}(\underline{\varphi}; \psi) \times (\mathbb{D}(\underline{\rho}_1; \varphi_1) \times_{\mathsf{Tight}(\mathbb{D})_1} \cdots \times_{\mathsf{Tight}(\mathbb{D})_1} \mathbb{D}(\underline{\rho}_n; \varphi_n)) \xrightarrow{\circ_{k_1, \dots, k_n}} \mathbb{D}(\underline{\rho}; \psi)$$

out of co-ends, where $|\underline{\rho_i}| = k_i$. Then $\mathbb{D}$ is a pro-representable VDC if and only if all such functions are isomorphisms.

PRO-REPRESENTABLE VDCS

Explication: Characterization of Pro-representable VDCs

In terms of pasting diagrams, a VDC $\mathbb{D}$ is pro-representable precisely when for any $N \geq 0$ and any partition $N = k_1 + \cdots + k_n$, N-multicells decompose as vertical pastings:



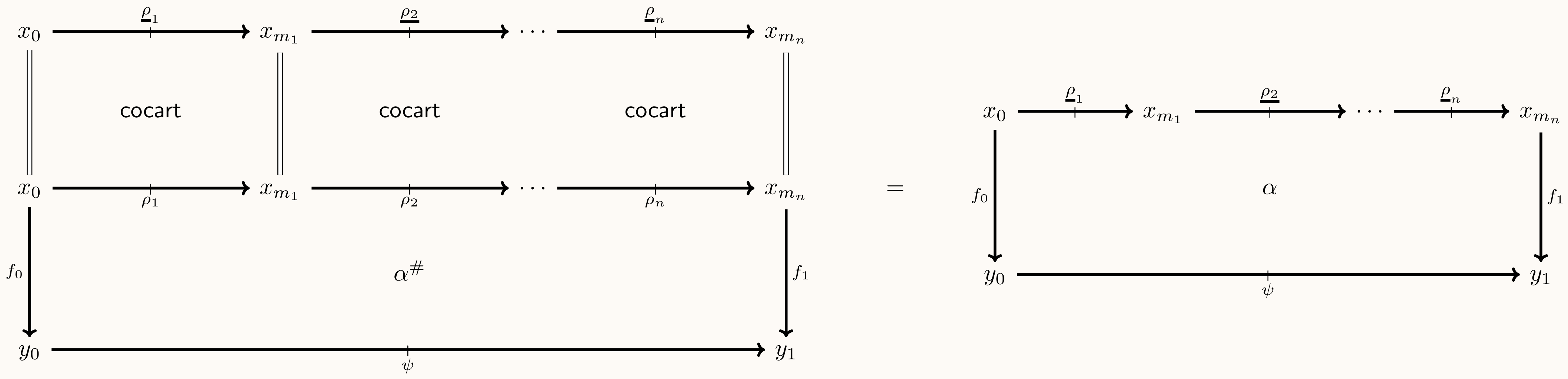
and any two decompositions are equivalent up to associativity of pasting with cells in the center of the decomposition

REPRESENTABLE VDCS

Corollary: Representable $\implies$ Pro-representable

Representable VDCs (i.e. pseudo-double categories) are pro-representable.

Proof Idea: Let $\mathbb{D}$ be a representable VDC. Then any cell admits a canonical decomposition



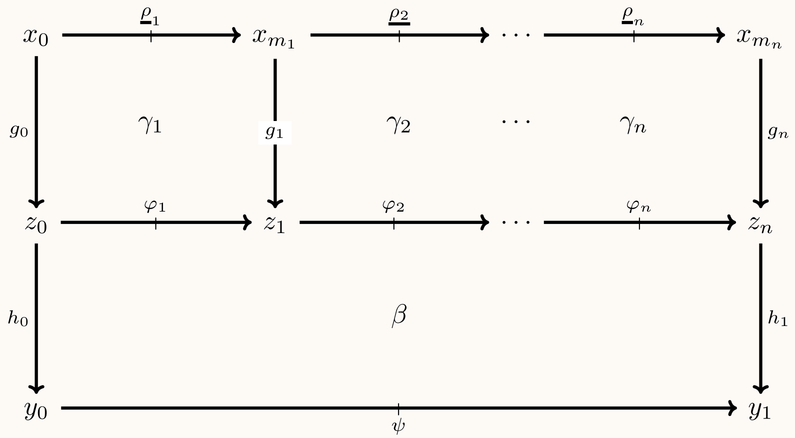
REPRESENTABLE VDCS

Corollary: Representable $\implies$ Pro-representable

Representable VDCs (i.e. pseudo-double categories) are pro-representable.

Proof Idea: (cont.)

Any other decomposition below left can be canonically factored through the composition cells:



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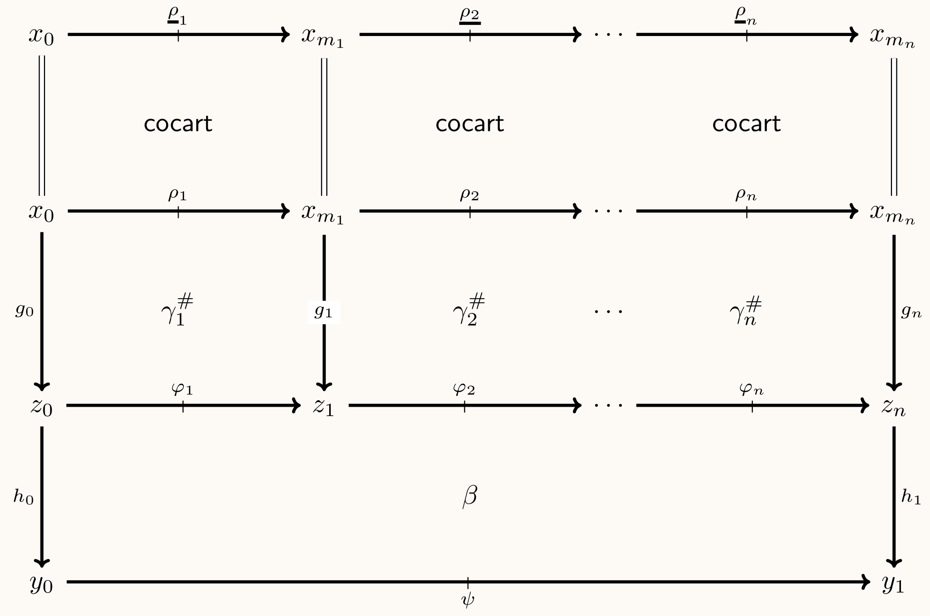
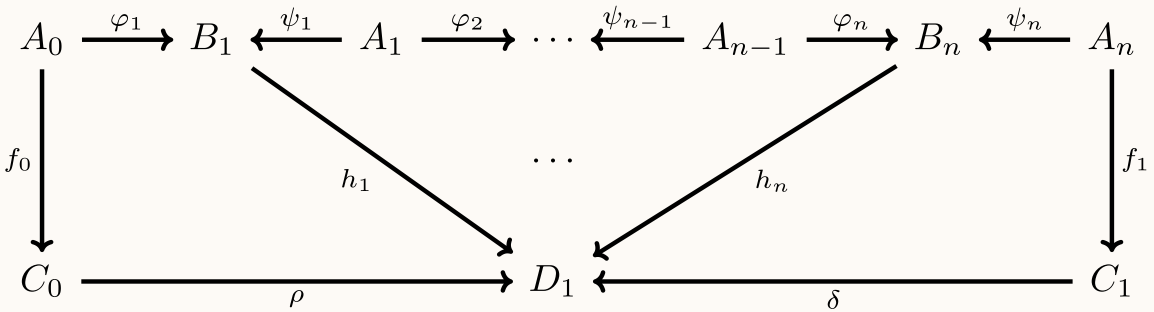


COSPANS ARE PRO-REPRESENTABLE

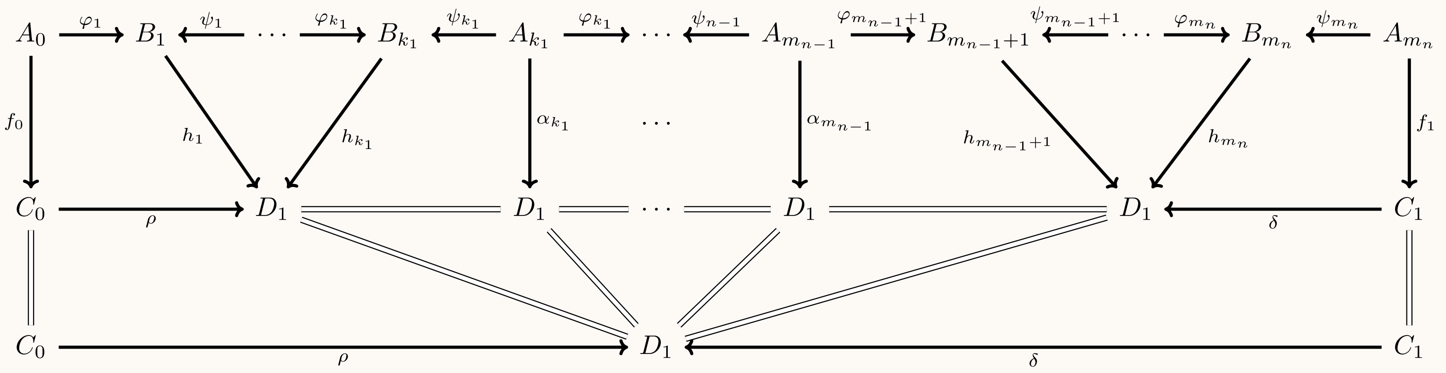
Proposition: Cospan VDCs are Pro-representable

For any category $\mathcal{E}$ the VDC $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is pro-representable, and it is representable if and only if $\mathcal{E}$ has finite pushouts.

Proof Idea: An arbitrary multicell:



admits a canonical decomposition:



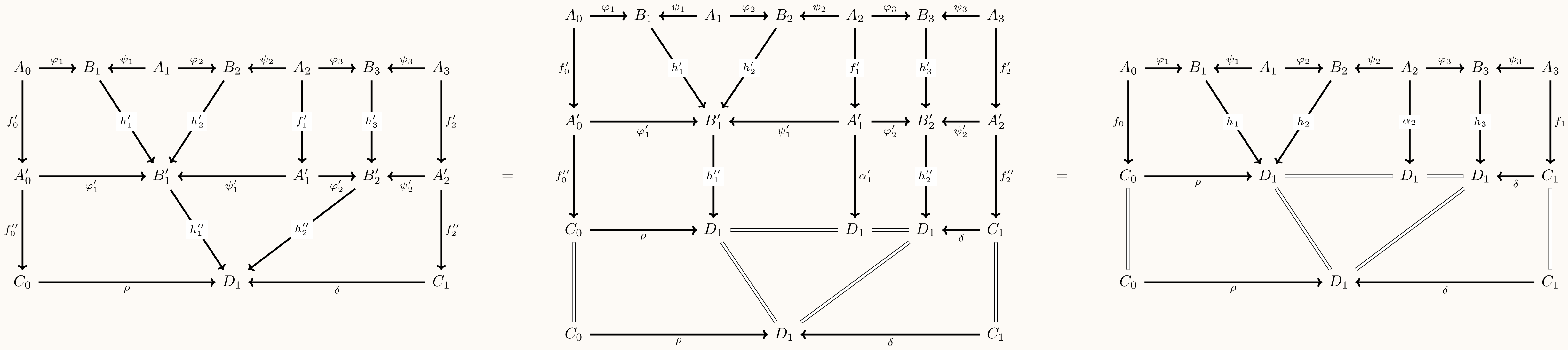
for any partition of n (the case where $k_1, k_n \geq 1$ is shown for simplicity).

COSPANS ARE PRO-REPRESENTABLE

Proposition: Cospan VDCs are Pro-representable

For any category $\mathcal{E}$ the VDC $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is pro-representable, and it is representable if and only if $\mathcal{E}$ has finite pushouts.

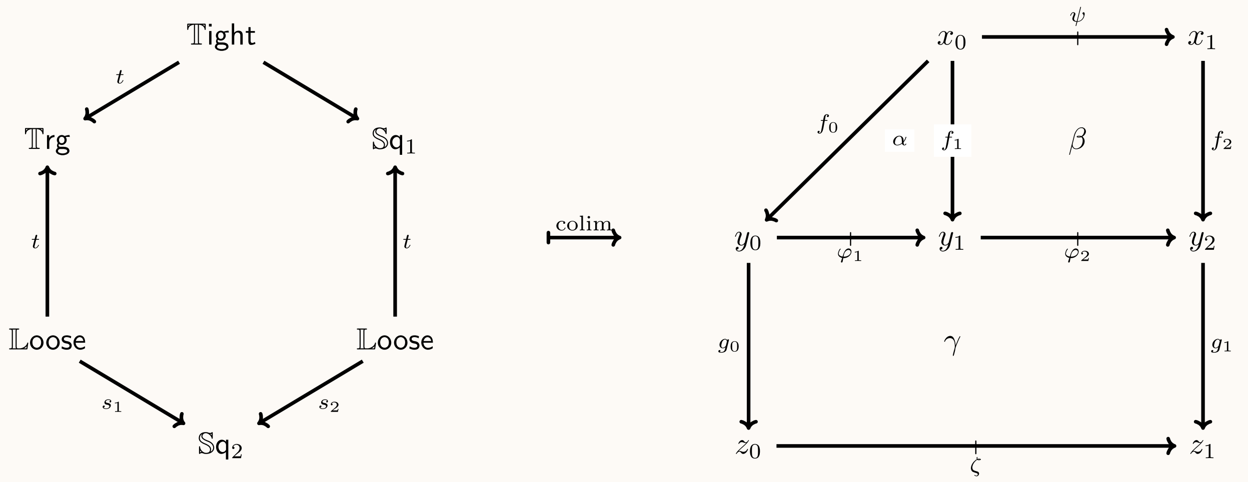
Proof Idea: For uniqueness consider the case where $n = 3, k_1 = 2, k_2 = 1$ as an example. Then an arbitrary decomposition, below left, can be seen to be equivalent to the canonical decomposition via sliding cells:



NON-PRO-REPRESENTABLE VDC

Non-example: Non-unital Walking Loose Arrow

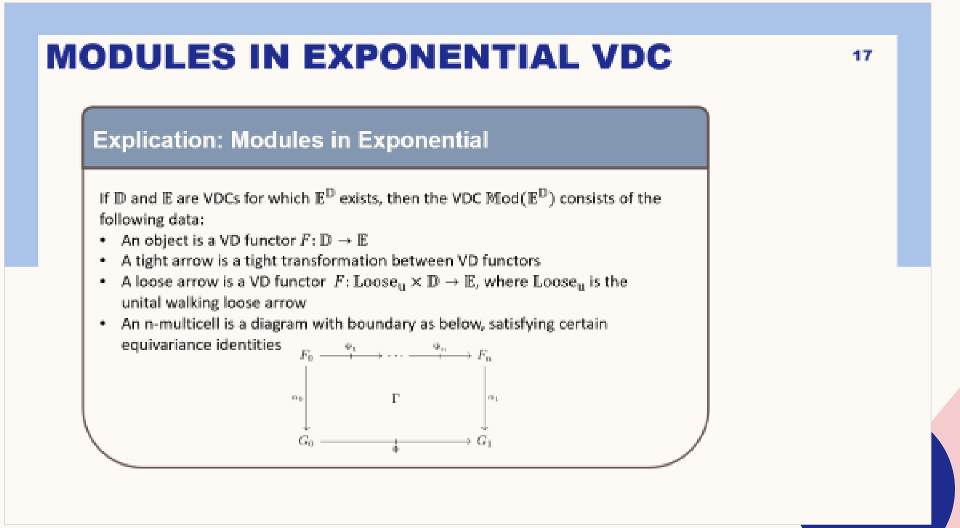
The VDC $\mathbb{L}\text{oose}$ consisting of two objects 0 and 1 and a single loose arrow $0 \nrightarrow 1$ is not pro-representable. Consider the diagram and colimit $\mathbb{C}$ in VDC depicted below:



Then $\mathbb{C} \times \mathbb{L}\text{oose}$ has two non-identity cells, while the VDC obtained by applying $- \times \mathbb{L}\text{oose}$ to the diagram before taking the colimit only has one.

APPENDIX

Modules in exponential VDCs:



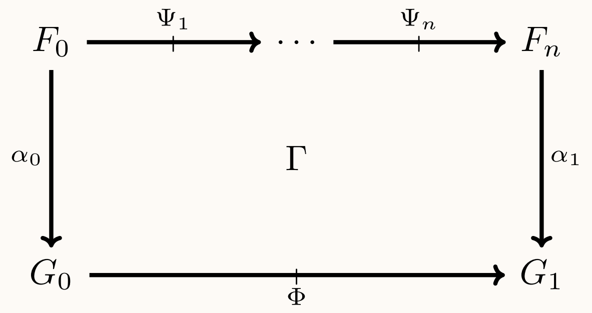
[Roadmap:]

MODULES IN EXPONENTIAL VDC

Explication: Modules in Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then the VDC $\mathbb{M}\mathrm{od}(\mathbb{E}^{\mathbb{D}})$ consists of the following data:

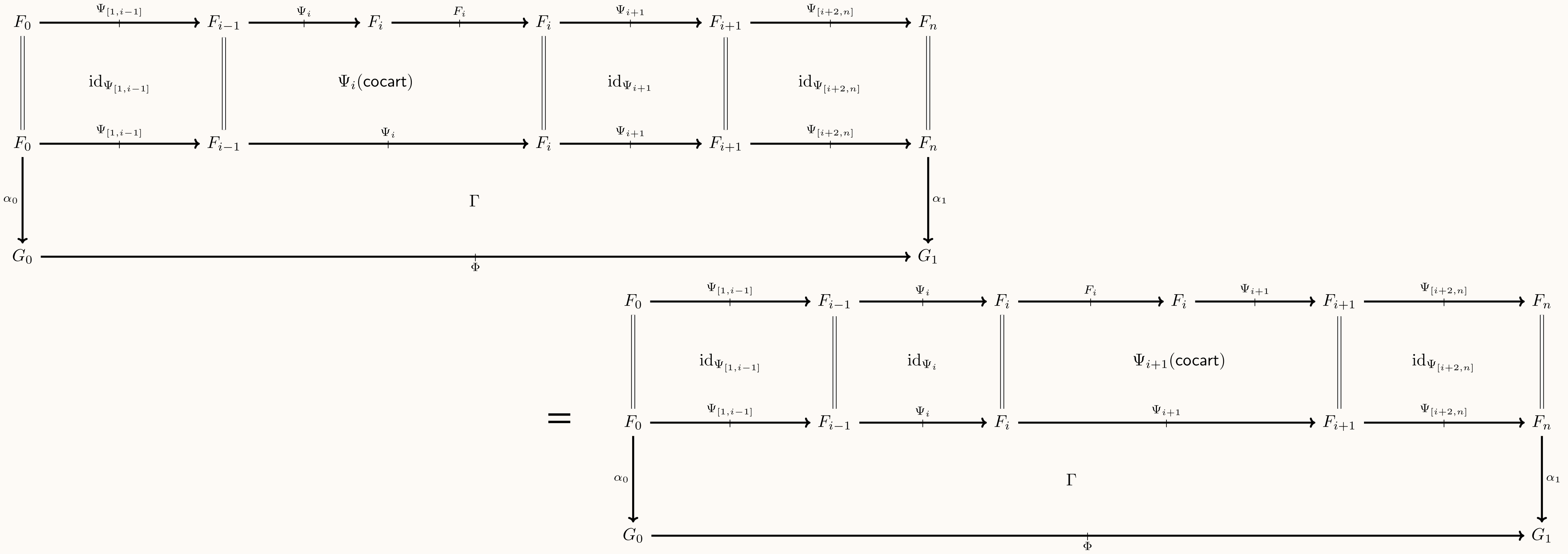
- An object is a VD functor $F\colon \mathbb{D} \to \mathbb{E}$
- A tight arrow is a tight transformation between VD functors
- A loose arrow is a VD functor $F\colon \mathbb{L}\mathrm{oose}_{\mathrm{u}} \times \mathbb{D} \to \mathbb{E}$, where $\mathbb{L}\mathrm{oose}_{\mathrm{u}}$ is the unital walking loose arrow
- An n-multicell is a diagram with boundary as below, satisfying certain equivariance identities



MODULES IN EXPONENTIAL VDC

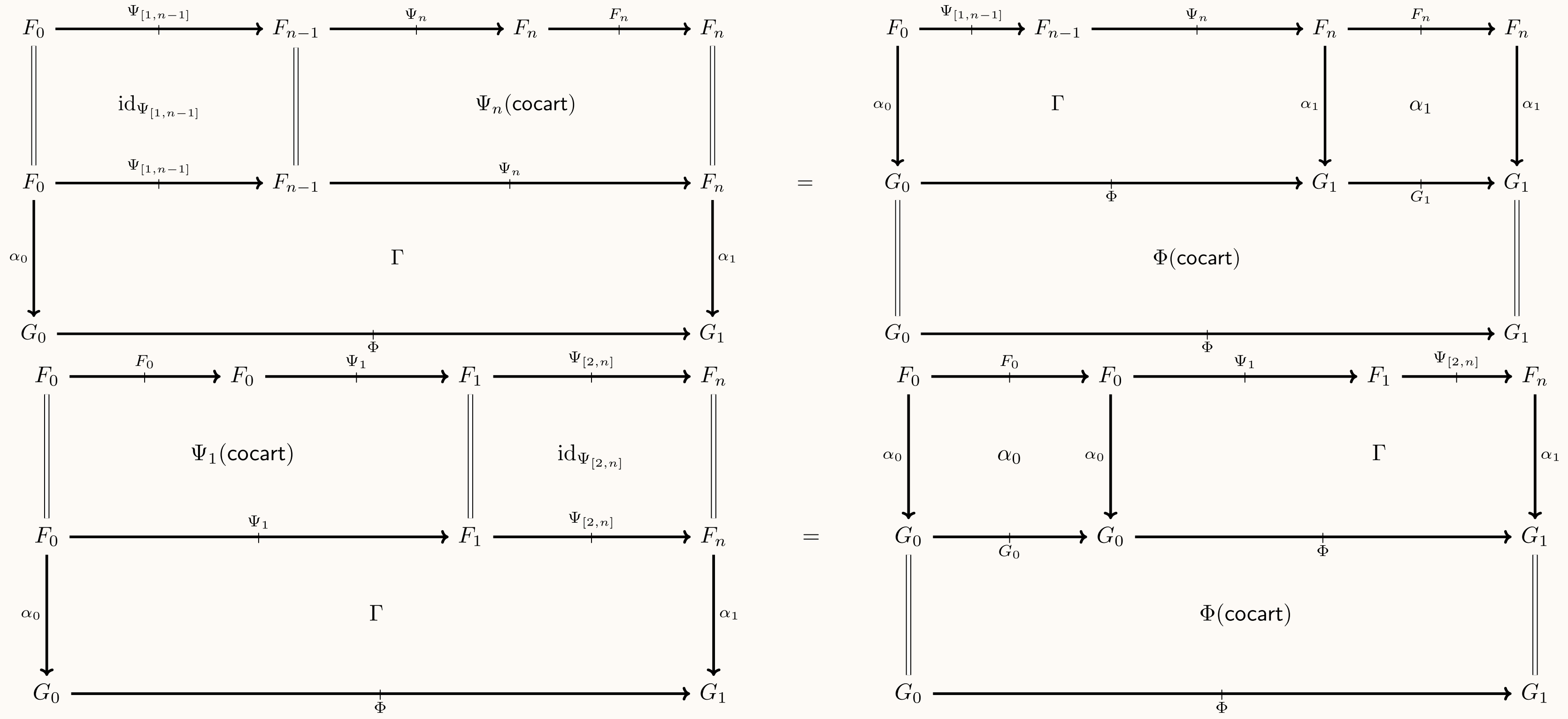
Explication: Modules in Exponential (Inner Equivariance)

Inner equivariance requires that the pasting diagrams below are equal for any $1 < i < n$:



Explication: Modules in Exponential (Outer Equivariance)

Outer equivariance requires we have the two pasting equalities below:



KEY TAKEAWAYS

- VDCs provide the necessary flexibility to characterize universal properties of double categorical constructions
- Exponentiable VDCs are those admitting essentially unique cell decompositions

FUTURE DIRECTIONS

- Characterize exponentiable maps between VDCs
- Determine what (co)limits and construction pro-representable VDCs are closed under
- Explore properties of enrichment using loose bimodules

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