# Twisted Ambidexterity via Spivak Data

## Introduction/Motivation

These notes are for a 30~40 minute talk on twisted ambidexterity in terms of Spivak data as appearing in Section 3.1-3.3 of the paper *Parametrised Poincare Duality and Equivariant Fixed Points Methods*[1], which was given as part of an Ambidexterity seminar at UIUC in Fall 2025. In the paper Hilman, Kirstein, and Kremer us the theory of parameterized $\infty$-categories to formulate and prove results about equivariant Poincare duality and its applications to fixed point methods. In the process of their work they provide a generalization of Bastiann Cnossen's theory of twisted ambidexterity[2] to the context of parameterized $\infty$-categories which are not necessarily locally presentable using a description analogous to the data for classical Poincare duality theorems.

Let's recall the classical Poincare duality picture:

> **Classical Poincare Duality**
>
> for a closed $d$-manifold $M$, there exists an infinite cyclic local coefficient system $\mathcal{O}$ on $M$ and a class $[M] \in H_d(M; \mathcal{O})$ such that the cap product with $[M]$ induces, for every local coefficient system $\eta$ on $M$, the isomorphisms
>
> $$[M] \cap - : H^*(M; \eta) \to H_*(M; \eta \otimes \mathcal{O})$$

Alternatively we can formulate the statement in terms of *local systems of spectra*. Let $M : M \to *$ be the unique map to a point, where we view $M \in \mathcal{S}$ as a space, forgetting its geometric structure. Then we get adjunctions

$$\mathsf{Sp}^M \;\; \begin{array}{c} \xrightarrow{\;M_!\;} \\ \perp \\ \xleftarrow{\;M^*\;} \\ \perp \\ \xrightarrow[\;M_*\;]{} \end{array} \;\; \mathsf{Sp}$$

For a local system $\xi \in \mathsf{Sp}^M$, we view $M_!\xi = \mathrm{colim}_M\, \xi$ (resp. $M_*\xi = \lim_M \xi$) as the homology (resp. cohomology) of $M$ with coefficients in the local system $\xi$. This becomes precise once we take homotopy groups if $\xi$ is a constant local system at a spectrum. For example, the **Spivak normal fibration** of a smooth manifold $M$ allows us to construct a $\otimes$-invertible local system $D_M \in \mathsf{pic}(\mathsf{Sp})^M \subseteq \mathsf{Sp}^M$. The *stable Pointryagin-Thom collapse map* then can be viewed as a map $c : \mathbb{S} \to M_! D_M$, which from our perspective that $M_! D_M$ represents $H_*(M; D_M)$ represents the *fundamental class* for $M$. We will describe later how we can form a cap product with the fundamental class as a morphism in $\mathsf{Fun}(\mathsf{Sp}^M, \mathsf{Sp})$:

$$c \cap - : M_*(-) \to M_!(- \otimes D_M)$$

and Poincare duality can be interpreted as the requirement that this transformation is an equivalence. This can be seen to be a reformulation of **twisted ambidexterity**, where $D_M$ is the *dualizing sheaf* of $M$. More generally, this can be done for any compact space $X$. We say a space is a **Poincare space** if it has a dualizing sheaf that takes values in invertible spectra $\mathsf{pic}(\mathsf{Sp})$. Consequently, all closed manifolds are Poincare spaces.

## Spivak Datums and Twisted Ambidexterity

Before defining the general theory let's motivate through classical Poincare duality.

> **Classical Poincare Duality**
>
> For a closed smooth manifold $M^d$, an embedding $M \hookrightarrow \mathbb{R}^N$ gives rise to a *collapse map*
>
> $$c : S^N \to \mathsf{Th}(\nu_{(M \hookrightarrow \mathbb{R}^N)})$$
>
> where $\nu_{(M \hookrightarrow \mathbb{R}^N)}$ is the normal bundle of the embedding. The stable homotopy type of this map doesn't depend on the choice of embedding, so we obtain a well-defined class $[c] \in H_N(\mathsf{Th}(\nu_{(M \hookrightarrow \mathbb{R}^N)})) \cong H_d(M; \mathcal{O}_\nu)$, where the isomorphism is the Thom isomorphism and $\mathcal{O}_\nu$ denotes the *orientation local system* for $\nu$. Classical Poincare duality says that
>
> $$[c] \cap - : H^k(M) \to H_{k-d}(M; \mathcal{O}_\nu)$$
>
> is an isomorphism.

This motivates the definition of *Spivak data*. Throughout we fix an $\infty$-topos $\mathcal{B}$ which we work internal to. For an object $\underline{X} \in \mathcal{B}$, we write $X : \underline{X} \to \underline{*}$ for the associated map to the terminal object.

## Spivak Datum

Let $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}$-category which admits $\underline{X}$-shaped colimits. A **$\underline{\mathcal{C}}$-Spivak datum** for $\underline{X}$ consists of

**(i)** an object $\xi \in \underline{\mathsf{Fun}}(\underline{X}, \underline{\mathcal{C}})$, called the **dualizing sheaf**;
**(ii)** a map $c : \mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ in $\underline{\mathcal{C}}$ called the **fundamental class** (or *collapse map*).

## Capping Map

Let $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}$-category which admits $\underline{X}$-shaped limits and colimits and satisfies the $\underline{X}$-projection formula (i.e. the Beck-Chevalley transformation $\mathsf{PF}_!^X : X_!(\xi \otimes X^*(-)) \to X_!\xi \otimes (-)$ is an equivalence for all $\xi \in \underline{\mathcal{C}}^{\underline{X}}$). For each $\underline{\mathcal{C}}$-Spivak datum $(\xi, c)$ on $\underline{X}$ we can construct a natural transformation

$$c \cap_\xi - : X_*(-) \xrightarrow{c\otimes -} X_!\xi \otimes X_*(-) \xleftarrow[\simeq]{\mathsf{PF}_!^X} X_!(\xi \otimes X^*X_*(-)) \xrightarrow{X_!(\mathrm{id}\otimes\epsilon)} X_!(\xi \otimes$$

which is a morphism in $\underline{\mathsf{Fun}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}})$.

## Fundamental Class from a Capping Map

Given a natural transformation $t : X_*(-) \to X_!(\xi \otimes -)$, and writing $\eta : \mathrm{id} \to X_*X^*$ for the adjunction unit, we obtain a **collapse map** as the composite

$$\mathsf{clps}_\xi(t) : \mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{\eta} X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{t} X_!(\xi \otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) \simeq X_!\xi$$

where in the last equivalence we are using the projection formula.

## Collapse map for Capping Map

There is an equivalence $\mathsf{clps}_\xi(c \cap_\xi -) \simeq c \in \mathsf{Map}_{\underline{\mathcal{C}}}(\mathbb{1}_{\underline{\mathcal{C}}}, X_!\xi)$.

**Proof.**
Consider the commutative diagram

$$
\begin{array}{ccccccc}
\mathbb{1}_{\underline{\mathcal{C}}} & \xrightarrow{\eta} & X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} & & & & \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c\otimes -} & & & & \\
X_!\xi & \xrightarrow[-\otimes\eta]{} & X_!\xi\otimes X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} & & & & \\
\uparrow{\scriptstyle \mathsf{PF}_!^X} & & \uparrow{\scriptstyle \mathsf{PF}_!^X} & & & & \\
X_!(\xi\otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) & \xrightarrow[X^*\eta]{} & X_!(\xi\otimes X^*X_*X^*\mathbb{1}_{\underline{\mathcal{C}}}) & \xrightarrow{\epsilon_{X^*}} & X_!(\xi\otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) & \xrightarrow[\simeq]{} & X_!\xi
\end{array}
$$

Then the top composite is precisely $\mathsf{clps}_\xi(c\cap_\xi -)$ while the bottom composite is equivalent to $c$ by the triangle identity.

□ ***

We will see that the capping maps produced from Spivak data often intertwine the left and right Beck-Chevalley transformations.

### Module Pushforwards from Multiplicative Basechanges

Suppose we have:

- Symmetric monoidal $\mathcal{B}$-categories $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$,
- A symmetric monoidal parameterized colimit-preserving functor $U:\underline{\mathcal{C}}\to\underline{\mathcal{D}}$ as well as a $\underline{\mathcal{C}}$-linear functor $F:\underline{\mathcal{C}}\to\underline{\mathcal{D}}$ using the $\underline{\mathcal{C}}$-linear structure on $\underline{\mathcal{D}}$ coming from $U$,
- A map $r:\underline{J}\to\underline{K}$ in $\mathbf{Cat}_{\mathcal{B}}$
- $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ admit left Kan extensions along $r$.

Then for $(\xi,c)$ a $\underline{\mathcal{C}}$-Spivak datum for $r$, we define

$$(\zeta,d):=\Big(U(\xi),U(c):\mathbb{1}_{\underline{\mathcal{D}}^{\underline{K}}}\to U(r_!\xi)\simeq r_!\zeta\Big)$$

as the associated $\underline{\mathcal{D}}$-Spivak datum for $r$. The data also gives symmetric monoidal functors $r^*:\underline{\mathcal{C}}^{\underline{K}}\to\underline{\mathcal{C}}^{\underline{J}}$ and $r^*:\underline{\mathcal{D}}^{\underline{K}}\to\underline{\mathcal{D}}^{\underline{J}}$, from which we can upgrade the functors $F^{\underline{K}}:\underline{\mathcal{C}}^{\underline{K}}\to\underline{\mathcal{D}}^{\underline{K}}$, $F^{\underline{J}}:\underline{\mathcal{C}}^{\underline{J}}\to\underline{\mathcal{D}}^{\underline{J}}$ to $\underline{\mathcal{C}}^{\underline{K}}$- and a $\underline{\mathcal{C}}^{\underline{J}}$-linear one, respectively.

The main examples of this structure that will be important is:

**(i)** When $F = U$ and the $\underline{\mathcal{C}}$-linear structure is given by the symmetric monoidality structure $UA \otimes U(-) \xrightarrow{\simeq} U(A \otimes -)$
**(ii)** When $\underline{\mathcal{C}} = \underline{\mathcal{D}}$, $U = \mathrm{id}_{\underline{\mathcal{C}}}$, and $F = A \otimes -$ for some fixed object $A \in \underline{\mathcal{C}}$, so that the $\underline{\mathcal{C}}$-linear structure on $F$ is the tautological one given by

$$\mathrm{id}(B) \otimes A \otimes - \simeq A \otimes B \otimes -$$

coming from the symmetric monoidal structure on $\underline{\mathcal{C}}$.

## Twisted Ambidexterity

Throughout let $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}$-category which admits $\underline{X}$-shaped limits and colimits and satisfies the $\underline{X}$-projection formula (for example this holds when $\underline{\mathcal{C}}$ is a presentably symmetric monoidal $\mathcal{B}$-category).

### Twisted Ambidexterity

A $\underline{\mathcal{C}}$-Spivak datum $(\xi, c)$ on $\underline{X}$ is **twisted ambidextrous** if the capping transformation $c \cap_\xi (-) : X_*(-) \to X_!(\xi \otimes -)$ is an equivalence.

Note that we also have a relative version of this definition, where for an object $\underline{Y} \in \mathcal{B}$ we can use the basechange adjunction $\pi_Y^* : \mathcal{B} \leftrightarrows \mathcal{B}_{/Y} : (\pi_Y)_*$.

### Twisted Ambidextrous Map

Consider a map $f : \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$-category $\underline{\mathcal{C}}$ such that the $\mathcal{B}_{/Y}$-category $(\pi_Y)^*\underline{\mathcal{C}}$ admits $f$ shaped limits and colimits that satisfy the $f$-projection formula.

A **$\underline{\mathcal{C}}$-Spivak datum** for $f$ is a $(\pi_Y)^*\underline{\mathcal{C}}$-Spivak datum for $f \in \mathcal{B}_{/Y}$. We say that such a Spivak datum exhibits $f$ as a **$\underline{\mathcal{C}}$-twisted ambidextrous map** if it exhibits $f \in \mathcal{B}_{/Y}$ as a $(\pi_Y)^*\underline{\mathcal{C}}$-twisted ambidextrous object.

We now show that in the presentable case twisted ambidextrous Spivak data are unique and that the notion of twisted ambidexterity is equivalent to that in[2-1].

### Twisted Ambidextrous Spivak Datum giving Adjunction

Let $(\xi, c)$ be a twisted ambidextrous $\underline{\mathcal{C}}$-Spivak datum for $\underline{X} \in \mathcal{B}$. The adjunction $X^* \dashv X_*$ induces an adjunction $X^* \dashv X_!(\xi \otimes -)$ whose unit is given by

$$\mathrm{id}(-) \xrightarrow{\mathrm{id}\otimes c} \mathrm{id}(-)\otimes X_!\xi \xleftarrow{\mathsf{BC}_!} X_!(X^*(-)\otimes\xi) = X_!(-\otimes\xi)\circ X^*(-)$$

**Proof.**
The fact that the equivalence $X_*(-) \simeq X_!(\xi\otimes -)$ induces an adjunction $X^* \dashv X_!(\xi\otimes -)$ follows from the uniqueness of adjoints (c.f. properties of adjunctions in infinity categories). For the description of the adjunction unit, observe that we have the commuting diagram

$$\begin{array}{ccccccc}
(-) & \xrightarrow{\mathrm{id}\otimes c} & (-)\otimes X_!\xi & \xleftarrow{\mathsf{BC}_!} & X_!(X^*(-)\otimes\xi) & & \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta\otimes X_!\xi} & & \downarrow{\scriptstyle X_!(X^*\eta\otimes\xi)} & \searrow{=} & \\
X_*X^*(-) & \xrightarrow[\mathrm{id}\otimes c]{} & X_*X^*(-)\otimes X_!\xi & \xleftarrow[\mathsf{BC}_!]{} & X_!(X^*X_*X^*(-)\otimes\xi) & \xrightarrow[X_!(\epsilon_{X^*}\otimes\xi)]{} & X_!(X^*(-)\otimes\xi)
\end{array}$$

where the bottom composite is the capping equivalence (pre-composed with $X^*(-)$) and the right triangle is by the triangle identity. Thus, the claimed map is compatible with the unit $\eta : \mathrm{id} \to X_*X^*$ under the capping equivalence $c\cap_\xi - : X_*(-) \xrightarrow{\simeq} X_!(\xi\otimes -)$, as desired.
□ ***

Note that if $\underline{\mathcal{C}} \in$ CAlg(Pr$^L_{\mathcal{B}}$) and if $X^* : \underline{\mathcal{C}} \to \underline{\mathcal{C}}^{\underline{X}}$ is an internal left adjoint in Mod$_{\underline{\mathcal{C}}}$(Pr$^L_{\mathcal{B}}$), then its right adjoint must be of the form $X_!(D_{\underline{X}}\otimes -)$ for a unique $D_{\underline{X}}$[2-2].

### The Presentable Case for Twisted Ambidexterity

Let $\underline{\mathcal{C}} \in$ CAlg(Pr$^L_{\mathcal{B}}$) be a presentably symmetric monoidal $\mathcal{B}$-category and $\underline{X} \in \mathcal{B}$.

**(1)** If $X^*$ is an internal left adjoint in Mod$_{\underline{\mathcal{C}}}$(Pr$^L_{\mathcal{B}}$) with right adjoint $X_!(D_{\underline{X}}\otimes -)$, then the unit map $c : \mathbb{1}_{\underline{\mathcal{C}}} \to X_!(X^*\mathbb{1}_{\underline{\mathcal{C}}}\otimes D_{\underline{X}}) \simeq X_!D_{\underline{X}}$ forms a $\underline{\mathcal{C}}$-twisted ambidextrous Spivak datum $(D_{\underline{X}}, c)$ for $\underline{X}$.
**(2)** If $(\xi, c)$ is a $\underline{\mathcal{C}}$-twisted ambidextrous Spivak datum for $\underline{X}$, then the map

$$(-) \xrightarrow{\mathrm{id}\otimes c} (-)\otimes X_!\xi \simeq X_!(X^*(-)\otimes\xi)$$

is the unit map of a $\underline{\mathcal{C}}$-linear adjunction $X^* \dashv X_!(-\otimes\xi)$

In particular, if $(\xi, c)$ and $(\xi', c')$ are twisted ambidextrous Spivak data, then there is an equivalence $\xi \simeq \xi'$ so that the composite $\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c} X_!\xi \simeq X_!\xi'$ is equivalent to $c'$.

Combining this result with Proposition 3.8 in[2-3] we see that $\underline{X}$ is $\underline{\mathcal{C}}$-twisted ambidextrous in the current sense if and only if it is in Bastiann Cnossen's sense, and in this case the twisted norm map $\mathsf{Nm}_{\underline{X}} : X_!(D_{\underline{X}} \otimes -) \to X_*(-)$ is an equivalence with inverse the map $\mathsf{Nm}_{\underline{X}}^{-1}(\mathbb{1}) \cap_{D_{\underline{X}}} (-)$.

**Twisted Ambidextrous Objects in Presentable Case**

Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$-category. An object of $\mathcal{B}$ is called **$\underline{\mathcal{C}}$-twisted ambidextrous** if it admits a (necessarily unique) twisted ambidextrous Spivak datum with coefficients in $\underline{\mathcal{C}}$.

For a twisted $\underline{\mathcal{C}}$-ambidextrous object $\underline{X} \in \mathcal{B}$, its associated Spivak datum is $(D_{\underline{X}}, \mathsf{Nm}_{\underline{X}}^{-1}(\mathbb{1}))$.

## Poincare Duality

The notion of Poincare duality comes from a special case of twisted ambidexterity where the local system for the Spivak datum takes values in tensor invertible objects.

**Poincare Spivak Datum**

A Spivak datum $(\xi, c)$ for $\underline{X}$ with coefficients in $\underline{\mathcal{C}}$ is **Poincare** if it is twisted ambidextrous and $\xi$ takes values in $\underline{\mathsf{Pic}}(\underline{\mathcal{C}})$.

In the case that $\underline{\mathcal{C}}$ is presentably symmetric monoidal, we say that the object $\underline{X}$ itself is **$\underline{\mathcal{C}}$-Poincare** if it is twisted $\underline{\mathcal{C}}$-ambidextrous and the unique twisted ambidextrous Spivak datum $(D_{\underline{X}}, c)$ is Poincare.

Again we can also define the relative version:

**Poincare Duality Maps**

Consider a map $f : X \to Y$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$-category $\underline{\mathcal{C}}$ such that the $\mathcal{B}_{/Y}$-category $(\pi_Y)^*\underline{\mathcal{C}}$ admits $f$-shaped limits and colimits and satisfies the $f$-projection formula. We say that a $\underline{\mathcal{C}}$-Spivak datum for $f$ exhibits $f$ as a **$\underline{\mathcal{C}}$-Poincare duality map** if it exhibits $f \in \mathcal{B}_{/Y}$ as a $(\pi_Y)^*\underline{\mathcal{C}}$-Poincare duality object.

### Poincare Duality from Higher Semiadditivity

**(1)** For any topos $\mathcal{B}$ and any symmetric monoidal $\mathcal{B}$-category $\underline{\mathcal{C}}$, the terminal object $\underline{*}$ has the tautological Poincare $\underline{\mathcal{C}}$-Spivak datum $(\mathbb{1}_{\underline{\mathcal{C}}}, \mathrm{id}_{\mathbb{1}_{\underline{\mathcal{C}}}})$.

**(2)** If $\underline{\mathcal{C}}$ is pointed, then the map $\underline{0} \to \underline{X}$ in $\mathcal{B}$ is $\underline{\mathcal{C}}$-Poincare

**(3)** If $\underline{\mathcal{C}}$ is semiadditive, then any finite fold map $\nabla : \coprod_{i=1}^n \underline{X} \to \underline{X}$ is $\underline{\mathcal{C}}$-Poincare

**(4)** More generally, Poincare spaces with trivial dualizing sheaf come from the theory of higher semiadditivity (c.f. Introduction to Semiadditivity and[3])

## Degree Theory

One of the important applications of the Spivak datum perspective is that it allows us to define a notion of degree for maps between Poincare spaces, which becomes an important computational invariant.

### Motivation: Classical Degree

Recall that given a map $f : X \to Y$ between closed connected manifolds of the same dimension, we can assign to it a degree if $f$ is compatible with the orientation behaviour of $X$ and $Y$. Namely, given an identification $\mathcal{O}_X \simeq f^*\mathcal{O}_Y$ of orientation local systems, the degree is given by the image of $[X]$ under $f_* : H_n(X; \mathcal{O}_X) \to H_n(Y; \mathcal{O}_Y) \cong \mathbb{Z}$.

In order to generalize this to our current setting we'll need versions of (co)homological functoriality:

### (Co)homological Functoriality

Consider a map $f : \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a $\mathcal{B}$-category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped (co)limits. We obtain transformations

$$\mathrm{BC}^f_! : X_! f^* \to Y_! \quad \text{and} \quad \mathrm{BC}^f_* : Y_* \to X_* f^*$$

of functors $\underline{\mathcal{C}}^{\underline{Y}} \to \underline{\mathcal{C}}$ coming from the left and right Beck-Chevalley transformations, respectively, associated to the commuting triangle $f^*Y^* \simeq X^*$. We call $\mathsf{BC}_!^f$ the **homological functoriality map** and $\mathsf{BC}_*^f$ the **cohomological functoriality map**.

### Degree of a map

Consider a map $f : \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric $\mathcal{B}$-category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped (co)limits and satisfies the $\underline{X}$- and $\underline{Y}$-projection formula. Suppose we have Spivak data $(\xi_{\underline{X}}, c_{\underline{X}})$ for $\underline{X}$ and $(\xi_{\underline{Y}}, c_{\underline{Y}})$ for $\underline{Y}$. A **$\underline{\mathcal{C}}$-degree datum** for $f$ is an equivalence

$$\alpha : \xi_{\underline{X}} \xrightarrow{\simeq} f^*\xi_{\underline{Y}}$$

in $\mathsf{Fun}_{\mathcal{B}}(\underline{X}, \underline{\mathcal{C}})$. We define the **$\underline{\mathcal{C}}$-degree** of $(f, \alpha)$ as the point $\mathsf{deg}_{\underline{\mathcal{C}}}(f, \alpha) \in \mathsf{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}})$ given by the composite

$$\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c_{\underline{X}}} X_!\xi_{\underline{X}} \xrightarrow{X_!\alpha} X_!f^*\xi_{\underline{Y}} \xrightarrow{\mathsf{BC}_!^f} Y_!\xi_{\underline{Y}}$$

We say that an equivalence $c_{\underline{Y}} \simeq \mathsf{deg}_{\underline{\mathcal{C}}}(f, \alpha)$ exhibits $f$ as a map of **$\underline{\mathcal{C}}$-degree one**.

### Commutative Monoid of Degrees

If the Spivak datum $(\xi_{\underline{Y}}, c_{\underline{Y}})$ is $\underline{\mathcal{C}}$-twisted ambidextrous, then the equivalence $c_{\underline{Y}} \cap_{\xi_{\underline{Y}}} \mathbb{1}_{\underline{\mathcal{C}}} : Y_*Y^*\mathbb{1}_{\underline{\mathcal{C}}} \simeq Y_!\xi_{\underline{Y}}$ endows $Y_!\xi_{\underline{Y}}$ with the structure of a commutative algebra in $\underline{\mathcal{C}}$. This gives $\mathsf{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}})$ the structure of a commutative monoid in $\mathcal{S}$ with unit $c_{\underline{Y}}$.

### $\mathcal{B} = \mathcal{S}$ with Presentably Symmetric Monoidal Coefficients $\mathcal{C}$

Let $f : X \to Y$ be a map of connected Poincare spaces of the same formal dimension $d$. We consider situations when a degree datum exists for the map $f$ with the Poincare Spivak data $(D_X, c_X)$ and $(D_Y, C_Y)$ for $X$, resp. $Y$.

**(1)** $\mathcal{C} = \mathsf{Mod}_{\mathbb{F}_2}$, a degree datum exists uniquely since $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{F}_2}) \simeq \mathbb{Z} \times B\mathsf{Aut}(\mathbb{F}_2) \simeq \mathbb{Z} \times *$ has contractible components.

**(2)** For $\mathcal{C} = \mathsf{Mod}_{\mathbb{Z}}$, writing $w_1(-)$ for the first Stiefel-Whitney class of a space, a degree datum exists if and only if $f^* w_1(Y) = w_1(X) \in H^1(X; \mathbb{F}_2)$. On homotopy groups the composite $X_! D_X \simeq X_! f^* D_Y \to Y_! D_Y$ then identifies with

$$f_* : H_{d+*}(X; \mathcal{O}_X) \to H_{d+*}(Y; \mathcal{O}_Y)$$

where $\mathcal{O}_X$ and $\mathcal{O}_Y$ denote the orientation local systems. The degree is then given by $f_*[X] \in H_d(Y; \mathcal{O}_Y) \cong \pi_0(\mathsf{Map}(\mathbb{1}_{\mathsf{Mod}_{\mathbb{Z}}}, Y_! D_Y))$, and agrees with the classical definition of the degree.

**(3)** In surgery theory, one is often provided with a **normal map** $f : X \to Y$ so that we have homotopy equivalences

$$(X \xrightarrow{f} Y \to BO \times \mathbb{Z}) \simeq (X \to BO \times \mathbb{Z})$$

$$(X \to BO \times \mathbb{Z} \xrightarrow{J} \mathsf{Pic}(\mathsf{Sp})) \simeq (X \xrightarrow{D_X} \mathsf{Pic}(\mathsf{Sp}))$$

and

$$(Y \to BO \times \mathbb{Z} \xrightarrow{J} \mathsf{Pic}(\mathsf{Sp})) \simeq (Y \xrightarrow{D_Y} \mathsf{Pic}(\mathsf{Sp}))$$

## References


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2. Bastiaan Cnossen. “Twisted Ambidexterity in Equivariant Homotopy Theory: Two Approaches.” Phd, Rheinische Friedrich-Wilhelms-Universität Bonn, 2024. https://hdl.handle.net/20.500.11811/11281 . ↩ ↩ ↩ ↩
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