Computing the Structure of Geometric Objects in the Langlands Program

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VXML Presentation

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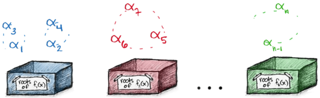
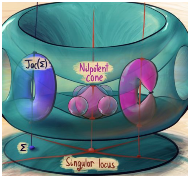
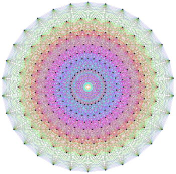
Motivation: The Langlands Program



(a) Algebraic Number Theory



(b) Algebraic Geometry



(c) Representation Theory

Figure: The Langlands Program: bridging fields

Vogan Variety: Preliminaries

Preliminary Data:

- A suitably "nice" matrix group G over a number system F
- A dual matrix group $\widehat{G}$ over $\mathbb{C}$
- An **infinitesimal parameter** $\lambda \in \widehat{G}$

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Infinitesimal Parameter: Informal

An infinitesimal parameter of $G(F)$ is a diagonal matrix $\lambda \in \widehat{G}$ of the form

$$\lambda = \mathrm{diag}(q^{e_0}, ..., q^{e_n})$$

where $e_0 \geq e_1 \geq \cdots \geq e_n \in \frac{1}{2}\mathbb{Z}$, and $q \in \mathbb{N}$ is related to the underlying number system F.

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Let $\lambda = \text{diag}(q^{e_0}, ..., q^{e_n}) \in \widehat{G}$ be an infinitesimal for $G(F)$.

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Vogan Variety

The Vogan Variety associated with λ is

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Toy Example (Geometry): Take $G(F) = \mathbf{GL}_2(F)$, $\widehat{G} = \mathbf{GL}_2(\mathbb{C})$, and $\lambda = \text{diag}(q^{1/2}, q^{-1/2})$. Then

$$V_\lambda = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_{2,2}(\mathbb{C}) : x \in \mathbb{C} \right\}$$

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Vogan Variety: Action

Group Action

V_λ has a natural group acting on it given by

$$H_\lambda = \{g \in \widehat{G} : \lambda g \lambda^{-1} = g\}$$

where $g \cdot M = gMg^{-1}$ for all $g \in H_\lambda, M \in V_\lambda$.

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$$H_\lambda = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in \mathbf{GL}_2(\mathbb{C}) : t_1, t_2 \in \mathbb{C}^\times \right\}$$

and the action is

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t_1 x/t_2 \\ 0 & 0 \end{pmatrix}$$

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Vogan Variety: Orbits

Orbit

An orbit of a point $x \in V_\lambda$ is the space

$$H_\lambda \cdot x = \{gxg^{-1} \in V_\lambda : g \in H_\lambda\}$$

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Toy Example (Orbits): Take $G(F) = \mathbf{GL}_2(F)$, $\widehat{G} = \mathbf{GL}_2(\mathbb{C})$, and $\lambda = \text{diag}(q^{1/2}, q^{-1/2})$. Then V_λ has two orbits:

$$C_0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \; C_1 = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_{2,2}(\mathbb{C}) : x \in \mathbb{C}^\times \right\}$$

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Local systems

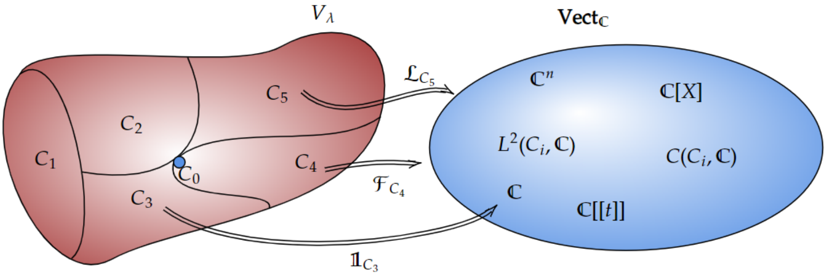

Figure: Local system on a Vogan

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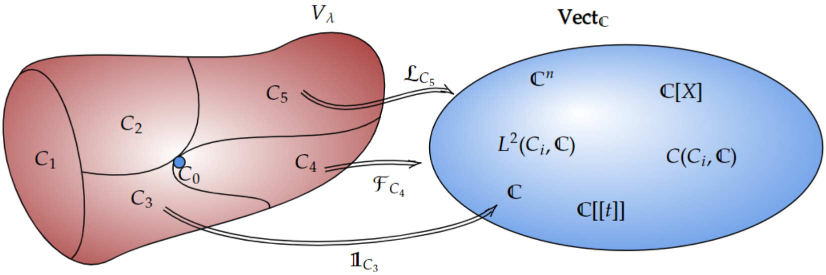

Figure: Local system on a Vogan

- Attached to a local system $\mathcal{L}_C$ on an orbit C is an object $IC(C, \mathcal{L}_C)$ on the whole Vogan variety.

$\mathbf{GL}_5(F)$ Case

- Consider the infinitesimal parameter $\lambda = \mathrm{diag}(q^1, q^0, q^0, q^0, q^{-1})$.

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$$V_\lambda = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : x_i, y_i \in \mathbb{C} \right\} \cong M_{1,3}(\mathbb{C}) \times M_{3,1}(\mathbb{C})$$

$$\cong \mathrm{Hom}(E_1, E_{q^1}) \times \mathrm{Hom}(E_{q^{-1}}, E_1)$$

$\mathbf{GL}_5(F)$ Case

- Our group is $H_\lambda \cong \mathbf{GL}_1(\mathbb{C}) \times \mathbf{GL}_3(\mathbb{C}) \times \mathbf{GL}_1(\mathbb{C})$
- $(g_0, g_1, g_2) \in H_\lambda$ acts on $(X_{1,0}, X_{2,1}) \in V_\lambda \cong \mathrm{Hom}(E_1, E_{q^1}) \times \mathrm{Hom}(E_{q^{-1}}, E_1)$ by

$$(g_0, g_1, g_2) \cdot (X_{1,0}, X_{2,1}) \cong (g_0 X_{1,0} g_1^{-1}, g_1 X_{2,1} g_2^{-1})$$

- We have five orbits:

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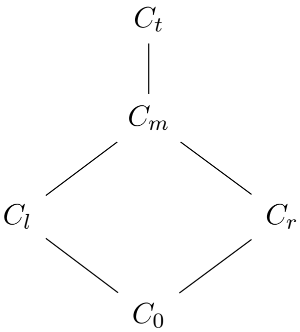

$\mathbf{GL}_5(F)$ Case

- The IC's on V_λ are

$$\{IC(C_0, \mathbb{1}_{C_0}), IC(C_l, \mathbb{1}_{C_l}), IC(C_r, \mathbb{1}_{C_r}), IC(C_m, \mathbb{1}_{C_m}), IC(C_t, \mathbb{1}_{C_t})\}$$

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Structure Table	(C_0)	(C_l)	(C_r)	(C_m)	(C_t)
$IC(C_0, \mathbb{1}_{C_0})$	$\mathbb{C}[0]$	0	0	0	0
$IC(C_l, \mathbb{1}_{C_l})$	$\mathbb{C}[3]$	$\mathbb{C}[3]$	0	0	0
$IC(C_r, \mathbb{1}_{C_r})$	$\mathbb{C}[3]$	0	$\mathbb{C}[3]$	0	0
$IC(C_m, \mathbb{1}_{C_m})$	$?$	$?$	$?$	$\mathbb{C}[5]$	0
$IC(C_t, \mathbb{1}_{C_t})$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$

Fixing Singularities: Resolutions

- We wish to find a smooth space $\widetilde{C_m}$ with a natural "nice" projection

$$\pi : \widetilde{C_m} \to \overline{C_m}$$

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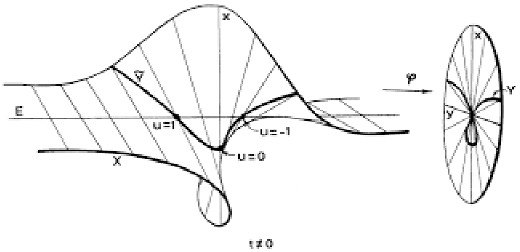

Figure: Resolution of Singularities through blow-up

Special Orthogonal Group and Symplectic Group

Special Orthogonal Group

For $n \in \mathbb{N}$, the special orthogonal group $SO(n)$ can be realized as

$$SO(n) = \{O \in \mathbf{GL}_n : O^T O = I_n,\ \det(O) = 1\}$$

Symplectic group

For $n \in \mathbb{N}$, the symplectic group Sp_{2n} can be realized as

$$Sp_{2n} = \{M \in \mathbf{GL}_{2n} : M^T \Omega M = \Omega\}$$

where most commonly $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

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- $\widehat{SO_{2n+1}(F)} = Sp_{2n}(\mathbb{C})$ and $\widehat{Sp_{2n}(F)} = SO_{2n+1}(\mathbb{C})$

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$SO_{2n+1}(\mathbb{C})$ Steinberg

- Let $\lambda = \text{diag}(q^n, q^{n-1}, ..., q^{-n}) \in SO_{2n+1}(\mathbb{C})$
- The Vogan variety is

$$V_\lambda^{SO} = \left\{ \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -x_2 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & -x_1 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in M_{2n+1}(\mathbb{C}) : x_1, ..., x_n \in \mathbb{C} \right\}$$

- The group acting on V_λ^{SO} is
$H_\lambda^{SO} = \{\text{diag}(t_1, ..., t_n, 1, 1/t_n, ..., 1/t_1) : t_1, ..., t_n \in \mathbb{C}^\times\}$

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$Sp_{2n}(\mathbb{C})$ Determinantal

- Let $\lambda = \text{diag}(q^{1/2}, ..., q^{1/2}, q^{-1/2}, ..., q^{-1/2}) \in Sp_{2n}(\mathbb{C})$ each occurring n times.
- The Vogan variety is

$$V_\lambda^{Sp} = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \in M_{2n}(\mathbb{C}) : A \in M_n(\mathbb{C}), A = A^T \right\}$$

- The group acting on V_λ^{Sp} is
$$H_\lambda^{Sp} = \left\{ \begin{pmatrix} X & 0 \\ 0 & (X^T)^{-1} \end{pmatrix} \in \mathbf{GL}_{2n}(\mathbb{C}) : X \in \mathbf{GL}_n(\mathbb{C}) \right\}$$

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Conclusions and Future Work

Results:

1. Symbolically represent Vogan varieties
2. Compute their orbit structure
3. Algorithm for resolving orbits in the case of $\mathbf{GL}_n$
4. Extend the algorithms to $SO(2n+1)$ and Sp_{2n} groups

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Future Work:

1. Continue expanding algorithms for computing the structure of IC's for $\mathbf{GL}_n, SO(n), Sp_{2n}$, and other classical groups
2. Use the tools and algorithms developed to study IC's for computing quantities such as the Ev functor

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Thank you for your time!

Any questions?

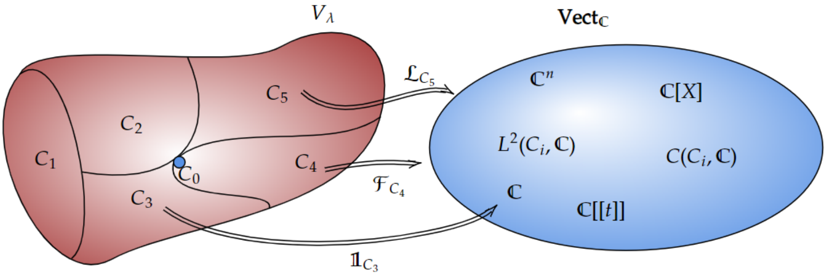

Figure: Local Systems

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