PACIFIC INSTITUTE FOR THE MATHEMATICAL SCIENCES VIRTUAL EXPERIMENTAL MATHEMATICS LAB (PIMS VXML) FINAL REPORT: p-atlas

Faculty Mentor: Clifton Cunningham
Graduate Mentor: Kristaps Balodis
Undergraduate Member: E Thompson

1 Introduction

This research project addresses an open problem in the Langlands Program relating to the classification of representations of certain p-adic groups. The Langlands Program consists of a vast number of wide-reaching conjectures worked on by mathematicians across the world. Its importance to the field of mathematics as a whole cannot be overstated, as it deals with the description of the "fundamental particles of arithmetic," L-Functions, and the tools used in its study connect fields ranging from number theory and representation theory to algebraic geometry and harmonic analysis. The work in this project is one of the many crucial problems in the program, and the methods used will be repurposed to simplify the approaches to the construction of examples for far more complex problems which are currently left unanswered.

Explicitly, the work in this project seeks to provide a first step at computational algorithms for use in the classification of representations of classical p-adic groups. This process begins with the historical characterization of p-adic representations into blocks parameterized by mathematical objects known as Langlands parameters. Tools from algebraic geometry are then brought into the picture through the translation provided by the p-adic Khazdhan Lusztig Hypothesis (pKLH). The pKLH relates the structure of special representations known as standard representations to the structure of moduli spaces of Langlands parameters, Vogan varieties. In particular, the relation is made through the analysis of certain objects defined on these Vogan varieties known as **simple perverse sheaves**. The full description of these objects is beyond the scope of this paper, but nonetheless, the structure of simple perverse sheaves on Vogan varieties is susceptible to the inductive algorithms we have constructed in this project, and hence are of primary focus.

1.1 The Initial Problem

The initial goal of the p-atlas was to begin the computational classification of representations of certain p-adic groups, in analogy with the Atlas developed in the real case. The initial restriction for this problem was the classification of certain representations of the group $\mathrm{GL}_n(F)$, where F is a p-adic field of residue cardinality $q_F \in \mathbb{N}$. In order to make this incredibly difficult task more tractable we leveraged the p-adic Khazdhan Luzstig Hypothesis in order to re-frame the problem geometrically. This involved the use of Vogan varieties, as characterized in the work of Cunningham, Fiori, Moussaoui, Mracek, and Xu [CFM+22]. Vogan varieties, being moduli spaces of Langlands parameters, are parameterized by objects known as **infinitesimal parameters** which classify Langlands parameters with the same Vogan variety.

Definition 1.1 (Infinitesimal Parameter: Informal)**.** An **infinitesimal parameter** of $\mathrm{GL}_n(F)$ is a diagonal matrix $\lambda \in \mathrm{GL}_n(\mathbb{C})$ of the form

$$\lambda = \mathrm{diag}(q_F^{e_0}, ..., q_F^{e_n})$$

where $e_0 \geq e_1 \geq \cdots \geq e_n \in \frac{1}{2}\mathbb{Z}$.

There is a much more general definition of infinitesimal parameters [CFM+22, p. 21], but this simplification is sufficient for the cases we consider, which are only unramified parameters of split classical groups. A Vogan variety associated with a given infinitesimal parameter is then characterized as follows.

Definition 1.2 (Vogan Variety: Informal)**.** The **Vogan variety** associated with an infinitesimal parameter $\lambda = \mathrm{diag}(q_F^{e_0}, ..., q_F^{e_n})$ in $\mathrm{GL}_{n+1}(\mathbb{C})$ is a matrix space defined by

$$V_\lambda := \left\{ M \in \mathrm{Lie}\,\mathrm{GL}_{n+1}(\mathbb{C}) : \lambda M \lambda^{-1} = q_F M \right\}$$

Attached to V_λ is a group

$$H_\lambda := \{ g \in \mathrm{GL}_{n+1}(\mathbb{C}) : \lambda g \lambda^{-1} = g \}$$

which acts by $g \cdot M := gMg^{-1}$.

In order to use the pKLH we must study the structure of simple perverse sheaves known as **intersection cohomology complexes** (ICs) on these Vogan varieties. Although the construction of ICs is beyond the scope of this paper, we may intuit them as being generated by **local systems** defined on the orbit closures of a given Vogan variety.

Definition 1.3 (Local System: Informal)**.** A **local system** $\mathcal{L}_C$ on an orbit closure $\overline{C}$ in a Vogan variety V_λ is an assignment of vector spaces to open sets in $\overline{C}$, such that the assignment is locally constant.

Within this framework, an IC built on an orbit C with local system $\mathcal{L}_C$ is denoted $IC(C, \mathcal{L}_C)$, and our original problem is restructured as the determination of the restrictions $IC(C, \mathcal{L}_C)|_{C'}$ of the ICs to each orbit C' in the Vogan variety V_λ. This reframed problem is inductive in nature, and is now receptive to algorithmic and computational approaches.

1.2 New Directions

While constructing an algorithm for determining the structure of ICs on a Vogan variety, we quickly hit the wall of orbit closures with singularities in our Vogan varieties. In the case of orbits with smooth closures we had well known results which allowed us to determine the restrictions of ICs on trivial local systems, $IC(C, \mathbb{1}_C)$ [Ach21]; a trivial local system can be thought of as one which assigns the one-dimensional vector space $\mathbb{C}$ to each open set in the orbit closure $\overline{C}$. However, in the case of non-singular orbit closures other techniques were required. In particular, for any orbit $C \subseteq V_\lambda$ with non-smooth closure $\overline{C}$, we required a smooth space $\widetilde{C}$ along with a suitably well-behaved covering map $\pi : \widetilde{C} \to \overline{C}$ so that we could extract the structure of $IC(C, \mathbb{1}_C)$ from the structure of $IC(\widetilde{C}, \mathbb{1}_{\widetilde{C}})$ which is susceptible to our known techniques since it is defined on a smooth space. This process is known as finding a **resolution of singularities** for our orbit closures. In order to reconcile this difficulty we underwent a thorough literature search in order to find such resolutions. In this process we were able to construct suitable resolutions in full generality for the case of $\mathrm{GL}_n(\mathbb{C})$ using previous work on Quiver varieties by Abaesis, Del Fra, and Kraft [ADFK81], as well as the classification of Vogan varieties in terms of Quiver varieties suggested by Zelevinsky's work [Zel81], and made more explicit by Cunningham and Ray [CR22].

Following this literature search and the creation of the algorithm we hit one more road block, which was the fact that our algorithm required a deep understanding of the fibers of the resolutions we constructed. The classification of these fibers proved to be in general beyond the scope of this project, although success was found for a number of infinite families of examples. Due to this roadblock we transitioned into exploring a new direction, which was the start of an algorithm for two other classical groups, namely the special orthogonal group of odd degree, $\mathrm{SO}_{2n+1}(\mathbb{C})$, and the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$. This investigation provided an opportunity to learn about these two groups along with their lie algebra structures, both of which have a number of applications in mathematics and physics. We continued investigating these two groups until the conclusion of the project.

2 Progress

Upon completion of this project we were able to fully classify the IC structure of a number of infinite families of Vogan varieties in the case of $\mathrm{GL}_n(\mathbb{C})$. Specifically, we were able to construct algorithmic results for infinitesimal parameters of the following forms:

I1. $\lambda = \mathrm{diag}\left(q_F^{(n-1)/2}, q_F^{(n-3)/2}, \ldots, q_F^{-(n-1)/2}\right)$, $V_\lambda \cong \mathbb{C}^n$

I2. $\lambda = \mathrm{diag}\left(\underbrace{q_F^{1/2}, \ldots, q_F^{1/2}}_{\ell}, \underbrace{q_F^{-1/2}, \ldots, q_F^{-1/2}}_{k}\right)$, $V_\lambda \cong M_{\ell,k}(\mathbb{C})$

I3. $\lambda = \mathrm{diag}\left(\underbrace{q_F^{1}, \ldots, q_F^{1}}_{\ell}, \underbrace{q_F^{0}, \ldots, q_F^{0}}_{k}, q_F^{-1}\right)$, $V_\lambda \cong M_{\ell,k}(\mathbb{C}) \times M_{k,1}(\mathbb{C})$

We also classified Vogan varieties which can be decomposed into products of these three forms. Examples of the produced IC tables for the (I1), (I2), and (I3) cases are given in Tables 1, 2, and 3.

Tab. 1: IC restriction (stalk) table for $\mathrm{GL}_3(\mathbb{C})$ with infinitesimal parameter for case (I1).

m_{geo}^{λ}	$\vert_{C_{0,0,0}}$	$\vert_{C_{1,0,0}}$	$\vert_{C_{0,1,0}}$	$\vert_{C_{1,1,1}}$
$IC(C_{0,0,0}, \mathbb{1}_{C_{0,0,0}})$	$\mathbb{C}[0]$	0	0	0
$IC(C_{1,0,0}, \mathbb{1}_{C_{1,0,0}})$	$\mathbb{C}[1]$	$\mathbb{C}[1]$	0	0
$IC(C_{0,1,0}, \mathbb{1}_{C_{0,1,0}})$	$\mathbb{C}[1]$	0	$\mathbb{C}[1]$	0
$IC(C_{1,1,1}, \mathbb{1}_{C_{1,1,1}})$	$\mathbb{C}[2]$	$\mathbb{C}[2]$	$\mathbb{C}[2]$	$\mathbb{C}[2]$

Tab. 2: IC restriction (stalk) table for $\mathrm{GL}_6(\mathbb{C})$ with infinitesimal parameter for case (I2) with $\ell = k = 3$.

m_{geo}^{λ}	$\vert_{C_0}$	$\vert_{C_1}$	$\vert_{C_2}$	$\vert_{C_3}$
$IC(C_0, \mathbb{1}_{C_0})$	$\mathbb{C}[0]$	0	0	0
$IC(C_1, \mathbb{1}_{C_1})$	$\mathbb{C}[5] \oplus \mathbb{C}[3] \oplus \mathbb{C}[1]$	$\mathbb{C}[5]$	0	0
$IC(C_2, \mathbb{1}_{C_2})$	$\mathbb{C}[8] \oplus \mathbb{C}[6] \oplus \mathbb{C}[4]$	$\mathbb{C}[8] \oplus \mathbb{C}[6]$	$\mathbb{C}[8]$	0
$IC(C_3, \mathbb{1}_{C_3})$	$\mathbb{C}[9]$	$\mathbb{C}[9]$	$\mathbb{C}[9]$	$\mathbb{C}[9]$

Tab. 3: IC restriction (stalk) table for $\mathrm{GL}_5(\mathbb{C})$ with infinitesimal parameter for case (I3) with $\ell = k = 2$.

m_{geo}^{λ}	$\vert_{C_{000}}$	$\vert_{C_{100}}$	$\vert_{C_{010}}$	$\vert_{C_{110}}$	$\vert_{C_{111}}$	$\vert_{C_{020}}$	$\vert_{C_{121}}$
$IC(C_{000}, \mathbb{1}_{C_{000}})$	$\mathbb{C}[0]$	0	0	0	0	0	0
$IC(C_{100}, \mathbb{1}_{C_{100}})$	$\mathbb{C}[2]$	$\mathbb{C}[2]$	0	0	0	0	0
$IC(C_{010}, \mathbb{1}_{C_{010}})$	$\mathbb{C}[3] \oplus \mathbb{C}[1]$	0	$\mathbb{C}[3]$	0	0	0	0
$IC(C_{110}, \mathbb{1}_{C_{110}})$	$\mathbb{C}[4] \oplus \mathbb{C}[2]$	$\mathbb{C}[4]$	$\mathbb{C}[4]$	$\mathbb{C}[4]$	0	0	0
$IC(C_{111}, \mathbb{1}_{C_{111}})$	$\mathbb{C}[5] \oplus \mathbb{C}[3]$	$\mathbb{C}[5] \oplus \mathbb{C}[3]$	$\mathbb{C}[5]$	$\mathbb{C}[5]$	$\mathbb{C}[5]$	0	0
$IC(C_{020}, \mathbb{1}_{C_{020}})$	$\mathbb{C}[4]$	0	$\mathbb{C}[4]$	0	0	$\mathbb{C}[4]$	0
$IC(C_{121}, \mathbb{1}_{C_{121}})$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$	$\mathbb{C}[6]$

Regarding our work in $\mathrm{SO}_{2n+1}(\mathbb{C})$ we were also able to determine the orbit and IC structure of case (I1), and demonstrated that it embeds equivariantly, i.e. preserving the group action, into a Vogan variety for $\mathrm{GL}_n(\mathbb{C})$, also of case (I1). On the other hand, for $\mathrm{Sp}_{2n}(\mathbb{C})$ we constructed the Vogan variety for case (I2) using symbolic tools in Sage, and found its orbit structure. However, the orbit closures in case (I2) were highly singular and hence required resolutions. Additionally, the (I2) case for $\mathrm{Sp}_{2n}(\mathbb{C})$ came with non-trivial ICs on all but the trivial orbit, which required multiple resolutions in order to parse out their structure. Thus, although we were able to construct a general resolution for this case, the study of the IC structure was too complicated to conclude before the end of this project.

2.1 Computational Work

In order to study the geometric structure of Vogan varieties, the first computational tools we constructed were related to symbolically generating the form of elements in a given Vogan variety. These tools were especially useful when studying the structure of the Vogan varieties for $\mathrm{SO}_{2n+1}(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$, as they were not as well studied as the Vogans for $\mathrm{GL}_n(\mathbb{C})$. For example, in the case of an infinitesimal parameter $\lambda = \mathrm{diag}\left(q_F^2, q_F^1, q_F^0, q_F^{-1}, q_F^{-2}\right)$ for $\mathrm{SO}_5(\mathbb{C})$, the code produced a general element of the

Vogan, M, and a general element of the group acting on the Vogan, g, of the forms

$$M = \begin{pmatrix} 0 & a_{12} & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & -a_{23} & 0 \\ 0 & 0 & 0 & 0 & -a_{12} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad g = \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & g_{22}^{-1} & 0 \\ 0 & 0 & 0 & 0 & g_{11}^{-1} \end{pmatrix}$$

The next important tool we created was one which generates the orbit lattice for a given Vogan variety. The lattice structure of orbits in a Vogan variety is induced by the natural partial ordering on orbits $C, C' \subseteq V_\lambda$, where $C' \leq C$ if and only if $C' \subseteq \overline{C}$.

Example 2.1. Consider the infinitesimal parameter $\lambda = \operatorname{diag}\left(q_F^1, q_F^1, q_F^0, q_F^0, q_F^{-1}, q_F^{-1}\right)$ for $\mathrm{GL}_6(\mathbb{C})$. Then using the code created we can generate the orbit lattice

$$\begin{tikzcd}
 & & C(2,2,2) & & \\
C(1,2,1) \arrow[urr] & & & & C(2,1,1) \arrow[ull] \\
 & & C(1,1,1) \arrow[ull] \arrow[urr] & & \\
C(0,2,0) \arrow[uu] & & C(1,1,0) \arrow[u] & & C(2,0,0) \arrow[uu] \\
C(0,1,0) \arrow[u] \arrow[urr] & & & & C(1,0,0) \arrow[u] \arrow[ull] \\
 & & C(0,0,0) \arrow[ull] \arrow[urr] & &
\end{tikzcd}$$

where $C(i,j,k)$ corresponds to an element

$$M = \begin{pmatrix} 0 & X_{10} & 0 \\ 0 & 0 & X_{21} \\ 0 & 0 & 0 \end{pmatrix} \in M_{6,6}(\mathbb{C}),\ X_{10}, X_{21} \in M_{2,2}(\mathbb{C})$$

where $\mathrm{rk}(X_{21}) = i, \mathrm{rk}(X_{10}) = j$, and $\mathrm{rk}(X_{10}X_{21}) = k$.

The main computational work that was done for this project was the creation of an inductive algorithm in Sage which computes the structure of the ICs attached to a Vogan variety for $\mathrm{GL}_n(\mathbb{C})$. We describe the overall approach of the algorithm in the following subsection.

2.1.1 Inductive Sage IC Algorithm

The algorithm begins by inputting an infinitesimal parameter of the form (I1), (I2), or (I3) (or a product thereof). Using a breadth-first search as well as a combinatorial tool known as a multisegment, all orbits in the Vogan variety with the given infinitesimal parameter are determined. In order to store the resulting IC restriction information of each orbit, a nested dictionary is created with the restrictions stored as polynomials. The powers of the polynomials indicate the shifts of the vector spaces in the restriction, and the coefficient in front of each term indicates the dimension of the vector space.

Once this initial setup is done the calculation of ICs is proceeded inductively from the smallest orbits to the largest orbits. Since there is not in general a total ordering on orbits, but only a partial ordering, the inductive process is done in terms of "levels" of the poset. Using well known theoretical results for ICs with smooth support the structures of ICs on orbits with non-singular closures are computed and stored. The computation for ICs supported on orbits with singular closures is more nuanced and requires a second inductive step.

In order to compute the IC for an orbit with a singular closure we rely on the Decomposition Theorem 2.5. Using this result requires the determination of multiplicities in Equation 2.1. This process is done using a descent from the orbit in question down the poset tree layer by layer. As we have performed already an initial induction up the poset tree with respect to determining IC restrictions, all orbits less than the singular orbit in question have their ICs fully determined. This allows for their use when determining multiplicities based off of restrictions to orbits. Note that in order to compute these restrictions we require the cohomologies of fibers above points, as described in Equation 2.2 in the Theoretical Work section. Once this process is completed all of the ICs and their restrictions are collected into a table which is formatted using LaTeX code. Examples to this effect are illustrated in Tables 1, 2, and 3.

2.2 Theoretical Work

Throughout this section fix an infinitesimal parameter λ of $\mathrm{GL}_n(\mathbb{C})$. The primary theoretical work needed for this project was related to the Sage IC Algorithm. To this end we first required a method of resolving singularities of orbit closures. Such a resolution of singularities for a space is defined, informally, as follows.

Definition 2.1 (Resolution of Singularities: Informal)**.** Let S be a singular variety. A **resolution of singularities** of S is a smooth variety $\widetilde{S}$ together with a **proper** and **birational** map $\pi : \widetilde{S} \to S$.

Intuitively the proper condition ensures that π behaves like a covering or projection onto S, while the birational condition restricts the size of $\widetilde{S}$, forcing $\widetilde{S}$ and S to have the same dimension. In general such resolutions are incredibly difficult to construct. However, using tools from the theory of Quiver varieties [ADFK81, pp. 410-411], we found a method of resolving orbit closures using special projective varieties known as partial flag varieties.

Definition 2.2 (Partial Flag Variety)**.** Let V be a vector space of dimension n and let $0 < e_1 < \cdots < e_k < n$ be a monotonic sequence of integers between 0 and n. Then the partial flag variety with index $(e_1, ..., e_k)$ is defined to be

$$\mathscr{F}(e_1, ..., e_k; n) = \left\{ (V_1, ..., V_k) \in \mathscr{P}(V)^k : V_1 \subseteq \cdots \subseteq V_k, \dim V_i = e_i, 1 \leq i \leq k \right\}$$

Elements of $\mathscr{F}(e_1, ..., e_k; n)$ are referred to as **flags**.

Then to resolve an orbit closure $\overline{C}$ for an orbit $C \subseteq V_\lambda$ we can extend points in $\overline{C}$ by solutions to projective equations. Before describing this result explicitly we require some notation and preliminary results. First, the work of Cunningham and Ray [CR22, p. 6, Lem 2.1] allows us to exhibit the identification

$$V_\lambda \cong \prod_{i=1}^{m} \mathrm{Hom}_{\mathbb{C}}(E_i, E_{i-1})$$

where $E_0, ..., E_m$ are the distinct neighboring eigenspaces of λ. Then an element in V_λ may be written in the form $(X_{1,0}, X_{2,1}, ..., X_{m,m-1})$. With this convention we define the following notation.

Definition 2.3 (Rank Triangle)**.** Let $X := (X_{1,0}, X_{2,1}, ..., X_{m,m-1}) \in V_\lambda$. We define the **rank triangle** for X to be the array $(r_{i,j})_{0\le j\le i\le m}$ where $r_{i,i} := \dim E_i$, and for $i > j$,

$$r_{i,j} := \mathrm{rk}(X_{j+1,j}X_{j+2,j+1}\cdots X_{i,i-1})$$

An essential result used in the classification Vogan varieties for our algorithm is that an orbit $C \subseteq V_\lambda$ is fully determined by the rank triangle of one of its elements [Rid22, p. 21, Prop 2.3.7]. If $(r_{i,j})_{0\le j\le i\le m}$ is the rank triangle for an orbit $C \subseteq V_\lambda$, we notate $\mathscr{F}_{i,C} = \mathscr{F}(r_{n,i}, ..., r_{i-1,i}; r_{i,i})$, which is a variety of flags in E_i.

Theorem 2.4 (Orbit Closure Resolution)**.** *If $C \subseteq V_\lambda$ is an orbit with rank triangle $(r_{i,j})_{0\le j\le i\le m}$, then the space*

$$\widetilde{C} := \left\{(X, F_0, ..., F_m) \in \overline{C} \times \prod_{i=0}^{m} \mathscr{F}_{i,C} : X_{i,i-1}\left(F_i^{(r_{j,i})}\right) \subseteq F_{i-1}^{(r_{j,i-1})}, 1 \le i \le j \le n\right\}$$

is smooth, and the projection $\pi : \widetilde{C} \to \overline{C}$ onto the first component is proper and birational, so $(\widetilde{C}, \pi)$ is a resolution of singularities. $F_i^{(r_{j,i})}$ is the subspace of dimension $r_{j,i}$ in the flag F_i.

A simple but important first example of this resolution in action is given below.

Example 2.2 (Determinantal $\mathrm{GL}_4(\mathbb{C})$)**.** Consider the infinitesimal parameter given by $\lambda = \mathrm{diag}\left(q_F^{1/2}, q_F^{1/2}, q_F^{-1/2}, q_F^{-1/2}\right)$. For this parameter $V_\lambda \cong M_{2,2}(\mathbb{C})$, and the orbits in the Vogan are precisely determined by rank. The orbit C_1 of rank 1 matrices has closure

$$\overline{C_1} \cong \{X \in M_{2,2}(\mathbb{C}) : \det X = 0\} = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_{2,2}(\mathbb{C}) : ad - bc = 0\right\}$$

which is singular at $a = b = c = d = 0$. In this case the rank triangle is given by $r_{0,0} = 2, r_{1,1} = 2$, and $r_{1,0} = 1$, so $\mathscr{F}_{0,C_1} = \{(0, U, E_0) : \dim U = 1\}$, and $\mathscr{F}_{1,C_1} = \{(0, E_1)\} \cong \{pt\}$. Truncating the notation we can write

$$\widetilde{C_1} = \left\{(X, U) \in \overline{C_1} \times \mathscr{F}_{0,C_1} : X_{1,0}(E_1) \subseteq U\right\}$$

This is a well-known resolution of what are known as **determinantal varieties** [Wey03, pp. 160-161].

The second primary theoretical tool required for the Sage IC algorithm was a deep understanding of the Decomposition Theorem [dCM07, p. 13].

Theorem 2.5 (Decomposition Theorem, Informal)**.** *Let $\pi : \widetilde{C} \to \overline{C}$ be a resolution of singularities of $C \subseteq V_\lambda$. Then*

$$\pi_! IC(\widetilde{C}, \mathbb{1}_{\widetilde{C}}) \cong \bigoplus_{i=-r(p)}^{r(p)} {}^{\mathfrak{p}}\mathcal{H}^i(R\pi_! IC(\widetilde{C}, \mathbb{1}_{\widetilde{C}}))[-i]$$

where $r(p) = \max_{C' \leq C}\{\dim C' + 2\dim \pi^{-1}(\{X\})_{X \in C'} - \dim C\}$ is the defect of semi-smallness. As each ${}^{\mathfrak{p}}\mathcal{H}^i(R\pi_! IC(\widetilde{C}, \mathbb{1}_{\widetilde{C}})) \cong \bigoplus_{C' \leq C} m_{i,C'} IC(C', \mathbb{1}_{C'})$, this can be rewritten as

$$\pi_! IC(\widetilde{C}, \mathbb{1}_{\widetilde{C}}) \cong \bigoplus_{i=-r(p)}^{r(p)} \bigoplus_{C' \leq C} m_{i,C'} IC(C', \mathbb{1}_{C'})[-i] \tag{2.1}$$

Although this theorem is extremely complex and dense, the final form that we were able to express it in was vital to the Sage IC algorithm, as it allowed us to gain information on ICs supported on singular orbit closures from information about an IC on a smooth space along with the ICs of smaller orbits. In particular, since we're dealing with a proper map out of a smooth space we have the description of stalks at points $X \in C' \leq C$ given by

$$(\pi_! IC(\widetilde{C}, \mathbb{1}_{\widetilde{C}})_X \cong H^\bullet(\pi^{-1}(\{X\}))[\dim C] \tag{2.2}$$

where we can think of a stalk at a point $X \in C'$ as a type of restriction down to C'. Thus, our problem reduces to the study of cohomologies of fibres above points in our Vogan variety for our resolution of singularities π. Although this was found to be in general a difficult task, in cases (I1), (I2), and (I3) the cohomologies are reasonable since the fibres may be described in terms of products of projective spaces.

Finally, one last essential result which I proved with respect to the decomposition theorem applied to the resolutions in Theorem 2.4 is the following.

Theorem 2.6. *Let V_λ be a Vogan variety with eigenspace $E_m, E_{m-1}, ..., E_0$, and let C be an orbit with singular closure. If $\pi : \widetilde{C} \to \overline{C}$ is the resolution of singularities given in Theorem 2.4, then $IC(C, \mathbb{1}_C)$ appears with multiplicity 1 and zero shift in $\pi_! IC(\widetilde{C}, \mathbb{1}_{\widetilde{C}})$.*

Theorem 2.6 is what allows us to use the decomposition theorem in finding the restrictions of $IC(C, \mathbb{1}_C)$ when $\overline{C}$ is singular.

We now briefly return to our example of the determinantal variety for $\mathrm{GL}_4(\mathbb{C})$ to illustrate these results.

Example 2.3 (Determinantal $\mathrm{GL}_4(\mathbb{C})$: Decomposition Theorem)**.** For the resolution $\pi : \widetilde{C_1} \to \overline{C_1}$, we have shown

$$\pi_! IC(\widetilde{C_1}, \mathbb{1}_{\widetilde{C_1}}) \cong IC(C_1, \mathbb{1}_{C_1})$$

Then since $\pi^{-1}(\{X\}) \cong \{pt\}$ for $X \in C_1$ we have

$$IC(C_1, \mathbb{1}_{C_1})|_{C_1} \cong \mathbb{C}[3]$$

where $3 = \dim C_1$. This aligns with previous theoretical results [Ach21, p. 543]. The interesting result occurs for restriction to $C_0 = \{0\}$, the zero orbit, as $\pi^{-1}(\{0\}) \cong \mathbb{P}^1$, the projective line, which has cohomology $H^\bullet(\mathbb{P}^1) \cong \mathbb{C}[0] \oplus \mathbb{C}[-2]$. Then

$$IC(C_1, \mathbb{1}_{C_1})|_{C_0} = \mathbb{C}[3] \oplus \mathbb{C}[1]$$

This fully describes the structure of $IC(C_1, \mathbb{1}_{C_1})$.

3 Future Directions

With the tools developed in this project we are able to classify the ICs of a number of Vogan varieties for $\mathrm{GL}_n(\mathbb{C})$. However, there are still an assortment of important cases yet to be fully resolved. Additionally, in consideration of the groups SO_{2n+1} and Sp_{2n}, as well as other classical matrix groups, a general algorithm for determining the structure of non-trivial ICs is still an open problem. Thus, there is still much work that can be done with respect to determining coverings of orbits, as well as how to dissect the information these coverings give in order to distinguish ICs attached to different local systems.

Another possible pathway for future work can be found in re-purposing the tools developed in this project so that they can be used to study other objects in the Langlands program. Importantly, the primary object of interest which may be elucidated by these algorithms is the Evs functor, which is currently being used to describe interesting packets of representations for p-adic groups such as GL_n.

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