The p-Atlas: Computing the Structure of Geometric Objects in the Langlands Program

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Introduction

The p-adic Kazdhan-Lusztig Hypothesis (pKLH) is a conjecture in the Langlands program which connects the structure of certain "standard" representations of a p-adic group G to geometric moduli spaces of **Langlands parameters**.

- Langlands parameters are mathematical objects which separate p-adic representations into structured blocks.
- The Langlands parameters corresponding to the same moduli space are characterized by a suitable diagonal matrix $\lambda \in \hat{G}$.
- The moduli space for $\lambda \in \hat{G}, q \in \mathbb{N}$:

$$V_\lambda = \{M \in \text{Lie } \hat{G} : \lambda M \lambda^{-1} = qM\}$$

known as a **Vogan variety**, with acting group

$$H_\lambda = \{g \in \hat{G} : \lambda g \lambda^{-1} = g\}$$

The goal of this project was to algorithmically classify the orbit structure of these spaces for certain classical groups and use the result to create algorithms for computing the structure of important mathematical objects known as **Intersection Cohomology Complexes (ICs).**

- ICs may be thought of as families of vector spaces attached to the orbits of the space.

In this poster we display results of our Sage and Python algorithms described in the report.

General Linear Group

A general Vogan variety decomposes in terms of linear maps between neighboring eigenspaces of λ:

$$V_\lambda \cong \prod_{i=1}^{n} \text{Hom}(E_i, E_{i-1})$$

The orbits are classified by the ranks of the component maps and their products [1].

Example: GL_4, $\lambda = \text{diag}(q^1, q^0, q^0, q^{-1})$,

Note $V_\lambda \cong M_{1,2}(\mathbb{C}) \times M_{2,1}(\mathbb{C})$. The orbits and their lattice structure are given below:

Tab 1. Orbit rank triangle structure.

Orbs	dim	rank-Δ
C_{000}	0	−1 0 1 1 2 1 0 0 0
C_{100}	2	−1 0 1 1 2 1 1 0 0
C_{010}	2	−1 0 1 1 2 1 0 1 0
C_{110}	3	−1 0 1 1 2 1 1 1 0
C_{111}	4	−1 0 1 1 2 1 1 1 1

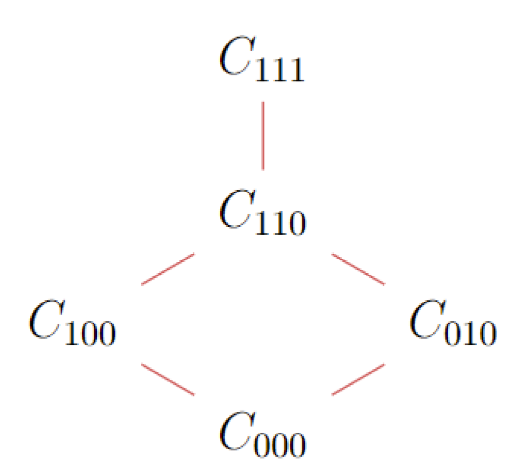


Fig 2. Orbit lattice for GL_4 example.

The only singular orbit is C_{110}, evidenced by its unique structure in the IC restriction table:

Tab 2. IC restriction table for GL_5 example.

m^λ_{geo}	$\vert_{C_{000}}$	$\vert_{C_{100}}$	$\vert_{C_{010}}$	$\vert_{C_{110}}$	$\vert_{C_{111}}$
$IC(C_{000}, \mathbb{1}_{C_{000}})$	$\mathbb{C}[0]$	0	0	0	0
$IC(C_{100}, \mathbb{1}_{C_{100}})$	$\mathbb{C}[2]$	$\mathbb{C}[2]$	0	0	0
$IC(C_{010}, \mathbb{1}_{C_{010}})$	$\mathbb{C}[2]$	0	$\mathbb{C}[2]$	0	0
$IC(C_{110}, \mathbb{1}_{C_{110}})$	$\mathbb{C}[3] \oplus \mathbb{C}[1]$	$\mathbb{C}[3]$	$\mathbb{C}[3]$	$\mathbb{C}[3]$	0
$IC(C_{111}, \mathbb{1}_{C_{111}})$	$\mathbb{C}[4]$	$\mathbb{C}[4]$	$\mathbb{C}[4]$	$\mathbb{C}[4]$	$\mathbb{C}[4]$

The algorithms developed in this project allow for similar computations for all λ's of the form:

- $\lambda = \text{diag}\left(q^{\frac{1}{2}}, \ldots, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, \ldots, q^{-\frac{1}{2}}\right)$,
- $\lambda = \text{diag}(q^1, \ldots, q^1, q^0, \ldots, q^0, q^{-1})$
- $\lambda = \text{diag}\left(q^{\frac{n-1}{2}}, \ldots, q^{\frac{-(n-1)}{2}}\right)$

as well as disjoint products thereof.

Special Orthogonal Group, $SO_{2n+1}(\mathbb{C})$

The methods for GL_n were extended to $\hat{G} = SO_{2n+1}(\mathbb{C})$. Embedding $SO_{2n+1}(\mathbb{C})$ into $GL_{2n+1}(\mathbb{C})$ allowed for the characterization [2]

$$V_\lambda^{SO} = \{M \in V_\lambda : JM + M^T J = 0\}$$

where J is a block antidiagonal matrix with block sizes determined by λ. For example, J may be

$$J = \begin{pmatrix} 0 & 0 & I_b \\ 0 & I_a & 0 \\ I_b & 0 & 0 \end{pmatrix}$$

Example: $SO_5, \lambda = \text{diag}(q^2, q^1, q^0, q^{-1}, q^{-2})$

$V_\lambda^{SO} \cong \mathbb{C} \times \mathbb{C}$, with an orbit determined by non-zero entries. All orbits are smooth and only trivial local systems exist making the IC structure simple.

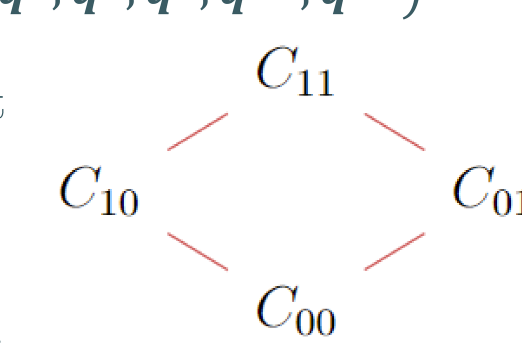


Fig 3. Orbit lattice for SO_5 example.

Tab 3. Restriction table for SO_5 example.

m^λ_{geo}	$\vert_{C_0}$	$\vert_{C_{1,0}}$	$\vert_{C_{0,1}}$	$\vert_{C_2}$
$IC(C_0, \mathbb{1}_{C_0})$	$\mathbb{C}[0]$	0	0	0
$IC(C_{1,0}, \mathbb{1}_{C_{1,0}})$	$\mathbb{C}[1]$	$\mathbb{C}[1]$	0	0
$IC(C_{0,1}, \mathbb{1}_{C_{0,1}})$	$\mathbb{C}[1]$	0	$\mathbb{C}[1]$	0
$IC(C_2, \mathbb{1}_{C_2})$	$\mathbb{C}[2]$	$\mathbb{C}[2]$	$\mathbb{C}[2]$	$\mathbb{C}[2]$

Symplectic Group, $Sp_{2n}(\mathbb{C})$

We also extended to $\hat{G} = Sp_{2n}(\mathbb{C})$. The same embedding applies but now with half the identity blocks in J negated. For example

$$J = \begin{pmatrix} 0 & -I_a \\ I_a & 0 \end{pmatrix}$$

Example: $Sp_6, \lambda = \text{diag}\left(q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}\right)$

The Vogan consists of symmetric matrices

$$V_\lambda = \left\{\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} : A \in M_{3\times 3}(\mathbb{C}), A^T = A\right\}$$

with orbits stratified by rank

$$C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0$$

Fig 4. Orbit lattice for Sp_6 example.

This is the first example that we find ICs attached to non-trivial local systems:

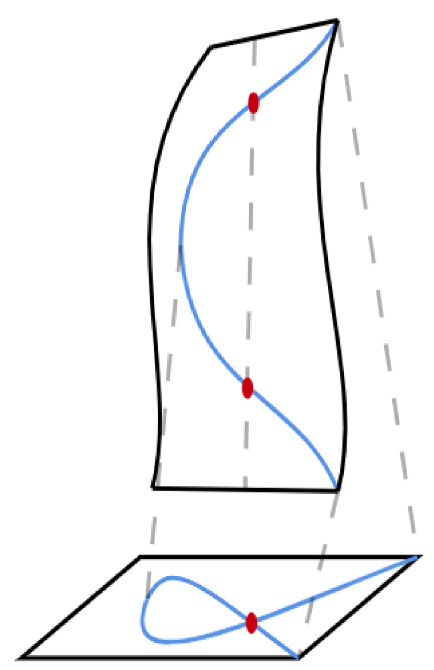
$$\{IC(C_0, \mathbb{1}_{C_0}), IC(C_r, \mathbb{1}_{C_r}), IC(C_r, \mathcal{E}_{C_r}) : 1 \le r \le 3\}$$

Due to this increased structure the determination of the IC restriction table is still a work in progress.

Future Work

Much future work is possible with this project, diverging into two main directions:

1. Continue developing methods for resolving singularities in orbits for classical groups for use in the inductive algorithm.
2. Repurpose the inductive algorithm to compute other important quantities such as evaluations of the Evs functor.



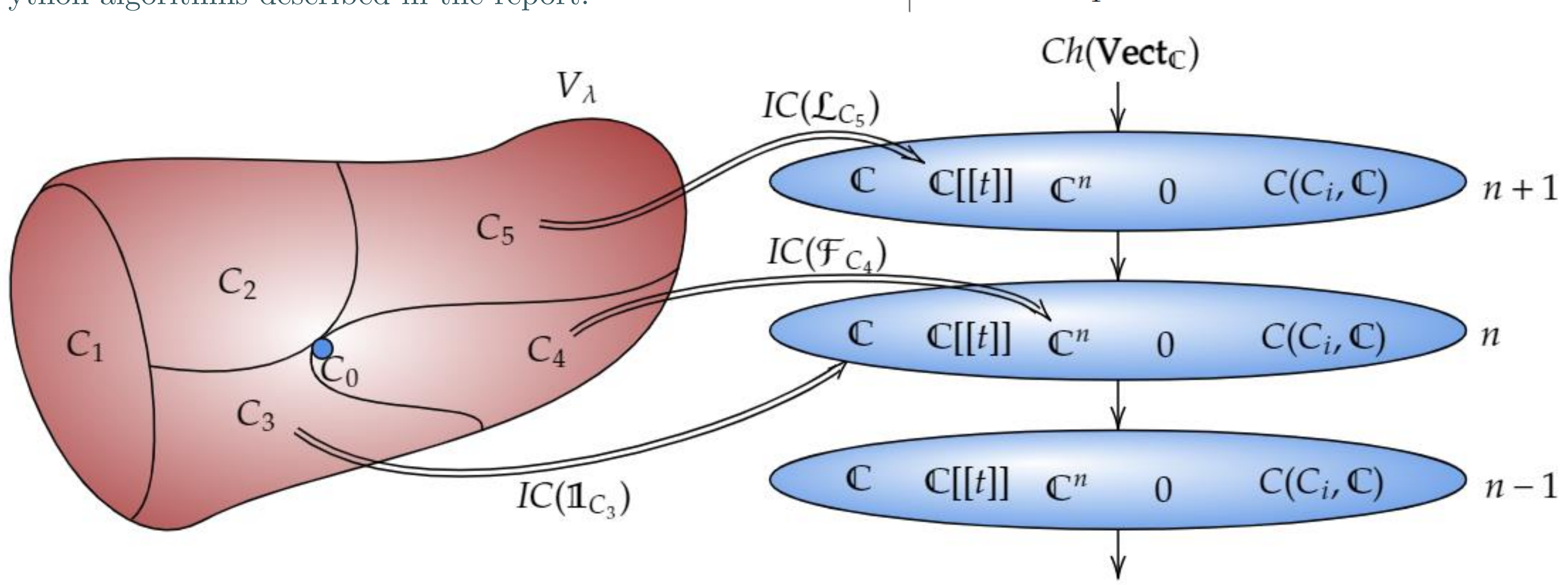


Fig 1. ICs on a Vogan variety visualized.

References

[1] Clifton Cunningham and Mishty Ray, Proof of vogan's conjecture on arthur packets: simple parameters of p-adic general linear groups, 2022

[2] Joel Benesh, Equivariant resolutions of singularities for orbits in generalized quiver varieties arising in the local langlands program for p-adic groups, Master's thesis, University of Lethbridge, 2022, p. 104.